05

When we come across such equations as $x^2 + 1 = 0$, $x^2 + 9 = 0$, we found ourselves unable to solve these equations, because $x^2 + 1 = 0$ gives $x^2 = -1$ or $x = \pm\sqrt{-1}$, as there is no number in the real number system, whose square is a negative number. Thus, to solve such type of problems, there is another number system called complex number system.

COMPLEX NUMBER

|TOPIC 1|

Introduction to Complex Numbers

CHAPTER CHECKLIST

- Introduction to Complex Numbers
- Algebra of Complex Numbers
- Conjugate, Modulus and Argand Plane of Complex Number

A number consisting of real number and imaginary number is called complex number. A complex number can be defined as a number of the form $a + ib$, where a and b are real numbers, is called a **complex number.**

e.g. $6 + 9i$, $-3 + 4i$ etc., are complex numbers.

Here, the symbol i is used to denote $\sqrt{-1}$ and it is called **iota.**

The complex number is generally denoted by z i.e. $z = a + ib$.

Complex number z can be represented in the form of order pair i.e. z can be represented as (a, b).

Knowledge Plus

Euler (1707-43) was the first mathematician, who introduced the symbol i (read as iota) for $\sqrt{-1}$ with property $i^2 + 1 = 0$ i.e. $i^2 = -1$. He also called this symbol as the imaginary unit.

REAL AND IMAGINARY PARTS OF A COMPLEX NUMBERS

Let $z = a + ib$ be a complex number, then a is called the **real part** and b is called the **imaginary part** of z and it may be denoted as **Re**(z) and **Im**(z), respectively.

e.g. If $z = 2 + 3i$, then $\text{Re}(z) = 2$ and $\text{Im}(z) = 3$.

PURELY REAL AND PURELY IMAGINARY COMPLEX NUMBERS

A complex number $z = a + ib$, is called **purely real**, if $b = 0$ i.e. $\text{Im}(z) = 0$ and is called **purely imaginary**, if $a = 0$ i.e. $\text{Re}(z) = 0$.

e.g. $z = 6$ is purely real and $z = 6i$ is purely imaginary.

ZERO COMPLEX NUMBER

A complex number is said to be zero, if its both real and imaginary parts are zero.

In other words, $z = a + ib = 0$, if and only if $a = 0$ and $b = 0$.

SET OF COMPLEX NUMBERS

The product set $R \times R$ consisting of the ordered pair of real number called the **set of real number**. The set of complex numbers is denoted by C and it is defined as

$$C = \{a + ib : a, b \in R\}$$

EXAMPLE |1| Find the real and imaginary parts of the following complex numbers.

(i) 7 (ii) $3i$

Sol. (i) Let $z = 7 = 7 + 0i$

Here, $\text{Re}(z) = 7$ and $\text{Im}(z) = 0$

(ii) Let $z = 3i = 0 + 3i$

Here, $\text{Re}(z) = 0$ and $\text{Im}(z) = 3$

Equality of Complex Numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal, if $a = c$ and $b = d$.

EXAMPLE |2| Find the real values of a and b, if

(i) $(3a - 6) + 2ib = -6b + (6 + a)i$

(ii) $(2a + 2b) + i(b - a) = -4i$

Equate the real and imaginary parts to get the required result.

Sol. (i) We have, $(3a - 6) + 2ib = -6b + (6 + a)i$

On equating real and imaginary parts, we get

$$3a - 6 = -6b \quad \text{...(i)}$$

and $$2b = 6 + a \quad \text{...(ii)}$$

Above equations can be rewritten as

$$3a + 6b = 6 \quad \text{...(iii)}$$

and $$a - 2b = -6 \quad \text{...(iv)}$$

On multiplying Eq. (iv) by 3 and then adding with Eq. (iii), we get

$$3a + 6b + 3a - 6b = 6 - 18$$

$\Rightarrow$ $6a = -12 \Rightarrow a = -2$

On substituting $a = -2$ in Eq. (iv), we get

$$-2 - 2b = -6$$

$\Rightarrow$ $-2b = -6 + 2$

$\Rightarrow$ $b = \frac{-4}{-2} = 2$

$\therefore$ $a = -2$ and $b = 2$

(ii) We have, $(2a + 2b) + i(b - a) = -4i$, which can be rewritten as

$$(2a + 2b) + i(b - a) = 0 - 4i$$

On equating real and imaginary parts, we get

$$2a + 2b = 0$$

$\Rightarrow$ $a + b = 0$ $[\because 2 \neq 0]$ ...(i)

and $b - a = -4$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$a + b + b - a = 0 - 4$$

$\Rightarrow$ $2b = -4 \Rightarrow b = -2$

On substituting $b = -2$ in Eq. (i), we get

$$a - 2 = 0 \Rightarrow a = 2 \therefore a = 2 \text{ and } b = -2$$

SOME IMPORTANT RESULT

We already know that, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real numbers a and b. This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$. But above result is not true for $a < 0, b < 0$, which can be explained as follows.

Let us consider, $i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)}$

[by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers]

$$= \sqrt{1} = 1$$

which is a contradiction to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$, if both a and b are negative real numbers.

Note

(i) If a is positive real number, then $\sqrt{-a} = \sqrt{a \times (-1)} = \sqrt{a} \times \sqrt{-1} = (\sqrt{a})i$

(ii) If any of a and b is zero, then, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$

EXAMPLE |3| Find the value of $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$.

Sol. We have, $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

$= \sqrt{25}\sqrt{-1} + 3\sqrt{4}\sqrt{-1} + 2\sqrt{9}\sqrt{-1}$

$= 5 \times i + 3 \times 2 \times i + 2 \times 3 \times i$ $[\because \sqrt{-1} = i]$

$= 5i + 6i + 6i = 17i$

EXAMPLE |4| Write the real and imaginary parts of the complex number $\sqrt{37} + \sqrt{-19}$.

Write the number $\sqrt{37} + \sqrt{-19}$ in the form $z = a + ib$ and compare, we get $\text{Re}(z) = a$ and $\text{Im}(z) = b$.

Sol. Let $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + \sqrt{19}\sqrt{-1}$

$\Rightarrow$ $z = \sqrt{37} + \sqrt{19}\,i$ $[\because \sqrt{-1} = i]$

$\therefore$ $\text{Re}(z) = \sqrt{37}$ and $\text{Im}(z) = \sqrt{19}$

INTEGRAL POWEROF i (IOTA)

I. POSITIVE INTEGRAL POWERS OF i

As we have seen, $i=\sqrt{-1}$. So, we can write the higher powers of i as follows

(i) $i^2=-1$

(ii) $i^3=i^2\cdot i=(-1)\cdot i=-i$

(iii) $i^4=(i^2)^2=(-1)^2=1$

(iv) $i^5=i^{4+1}=i^4\cdot i=1\cdot i=i$

(v) $i^6=i^{4+2}=i^4\cdot i^2=1\cdot i^2=-1$

⋮ ⋮ ⋮ ⋮

While evaluating i^n for $n>4$, we are writing n as $4q+r$ for some $q,r\in N$ and $0\le r\le 3$. So, in order to compute i^n for $n>4$, write $i^n=i^{4q+r}$ for some $q,r\in N$ and $0\le r\le 3$. Then, $i^n=i^{4q}\cdot i^r=(i^4)^q\cdot i^r=(1)^q\cdot i^r=i^r$

e.g. $i^{17}=i^{4\times4+1}=i^{4\times4}\cdot i=(i^4)^4\cdot i=1\cdot i=i$

i^4 is defined as 1.

Note In general for any integer k, $i^{4k}=1$, $i^{4k+1}=i$, $i^{4k+2}=-1$ and $i^{4k+3}=-i$

II. NEGATIVE INTEGRAL POWERS OF i

Negative integral powers of i can be evaluated as follows

(i) $i^{-1}=\frac{1}{i}=\frac{1}{i}\times\frac{i}{i}$ [multiply numerator and denominator by i]

$=\frac{i}{i^2}=\frac{i}{(-1)}=-i$ $[\because i^2=-1]$

(ii) $i^{-2}=\frac{1}{i^2}=\frac{1}{(-1)}=-1$

(iii) $i^{-3}=\frac{1}{i^3}=\frac{1}{i^3}\times\frac{i}{i}$ [multiplying numerator and denominator by i]

$=\frac{i}{(i^4)}=\frac{i}{(1)}=i$

(iv) $i^{-4}=\frac{1}{i^4}=\frac{1}{1}=1$

In order to compute i^{-n} for $n>4$, first write $i^{-n}=\frac{1}{i^n}=\frac{1}{i^{4q+r}}$ for some $q,r\in N$ and $0\le r\le 3$. Then, evaluate i^{4q+r}. Further, use above four negative integral powers of i.

e.g. $i^{-15}=\frac{1}{i^{15}}=\frac{1}{i^{4\times3+3}}=\frac{1}{i^3}$ $[\because i^{4q+3}=i^3]$

$=\frac{1}{i^3}\times\frac{i}{i}=\frac{i}{i^4}=\frac{i}{1}=i$ $[\because i^4=1$

EXAMPLE |5| Find the value of

(i) i^{37} (ii) i^{-30} (iii) $\frac{1}{i^7}$

Sol. (i) We have, $i^{37}=(i)^{36+1}=(i)^{4\times9}i$

$=(i^4)^9\cdot i=(1)^9\cdot i=i$ $[\because i^4=1]$

(ii) We have, $i^{-30}=\frac{1}{i^{30}}$

Now, $i^{30}=(i)^{4\times7+2}=(i^{4\times7})\,i^2=(i^4)^7\,(-1)$ $[\because i^2=-1]$

$=(1)^7\,(-1)=-1$ $[\because i^4=1]$

$\Rightarrow i^{-30}=\frac{1}{(-1)}=-1$

(iii) We have, $\frac{1}{i^7}=\frac{1}{(i)^{4+3}}=\frac{1}{i^4\cdot i^3}$

$=\frac{1}{1\cdot(-i)}$ $[\because i^4=1$ and $i^3=-i]$

$=\frac{1}{-i}\times\frac{i}{i}=\frac{i}{-i^2}=\frac{i}{-(-1)}=i$ $[\because i^2=-1]$

EXAMPLE |6| Express the following in the form of $a+ib$, where $a,b\in R$.

(i) i^{103} (ii) $(-\sqrt{-1})^{4x+3}$ (iii) $\left(i^{29}+\frac{1}{i^{29}}\right)$

Sol. (i) $i^{103}=i^{25\times4+3}=(i^4)^{25}\cdot i^3=(1)^{25}\cdot(-i)$

$[\because i^4=1$ and $i^3=-i]$

$=-i=0-i$

(ii) $(-\sqrt{-1})^{4x+3}=(-i)^{4x+3}=(-i)^{4x}\cdot(-i)^3$

$=i^{4x}(-i^3)=(i^4)^x(-(-i))$ $[\because i^3=-i]$

$=(1)^x(i)$ $[\because i^4=1]$

$=i=0+i$

(iii) $i^{29}+\frac{1}{i^{29}}=\frac{i^{29}\cdot i^{29}+1}{i^{29}}=\frac{(i^2)^{29}+1}{i^{29}}$

$=\frac{(-1)^{29}+1}{i^{29}}=\frac{-1+1}{i^{29}}=0=0+0i$

EXAMPLE |7| Find the value of $\left[i^{19}+\left(\frac{1}{i}\right)^{25}\right]^2$.

Sol. $\left[i^{19}+\left(\frac{1}{i}\right)^{25}\right]^2=\left[i^{4\times4+3}+\frac{1}{i^{4\times6+1}}\right]^2$

$=\left[(i^4)^4(i)^3+\frac{1}{(i^4)^6 i}\right]^2=\left[(1)^4(i)^3+\frac{1}{(1)^6\,i}\right]^2$

$[\because i^4=1$ and $i^3=-i]$

$=\left(-i+\frac{1}{i}\right)^2=\left(-i+\frac{i}{i^2}\right)^2=\left(-i+\frac{i}{i\times i}\right)^2=\left(-i+\frac{i}{-1}\right)^2$

$[\because i^2=-1]$

$=(-i-i)^2=(-2i)^2=4i^2=-4$ $[\because i^2=-1]$

EXAMPLE |8| Show that
$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \forall n \in N.$$

Sol. LHS $= i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n + i^n \cdot i + i^n \cdot i^2 + i^n \cdot i^3$

$= i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i)$ $\quad [\because i^2 = -1, i^3 = i^2 \cdot i = -i]$

$= i^n(0) = 0 =$ RHS **Hence proved.**

EXAMPLE |9| Evaluate $\dfrac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$.

Sol. Consider the given expression,

$$\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$$

$$= \frac{i^{584+8} + i^{584+6} + i^{584+4} + i^{584+2} + i^{584}}{i^{574+8} + i^{574+6} + i^{574+4} + i^{574+2} + i^{574}}$$

$$= \frac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)}$$

$$= \frac{i^{584}}{i^{574}} = i^{584-574} = i^{10}$$

$$= i^{4 \times 2 + 2} = (i^4)^2 \cdot i^2$$

$= (1)^2 \cdot i^2 = -1$ $\quad [\because i^4 = 1 \text{ and } i^2 = -1]$

EXAMPLE |10| What is the value of $\dfrac{i^{4x+1} - i^{4x-1}}{2}$? **[NCERT Exemplar]**

Sol. Consider, $\dfrac{i^{4x+1} - i^{4x-1}}{2} = \dfrac{i^{4x} \cdot i - i^{4x} \cdot i^{-1}}{2} = \dfrac{i - \frac{1}{i}}{2}$ $\quad [\because i^{4x} = 1]$

$= \dfrac{i^2 - 1}{2i} = \dfrac{-2}{2i}$ $\quad [\because i^2 = -1]$

$= \dfrac{-1}{i} = \dfrac{-i}{i^2}$

$= \dfrac{-i}{-1} = i$ $\quad [\because i^2 = -1]$

EXAMPLE |11| Find the real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real. **[NCERT Exemplar]**

Sol. $3i^3 - 2ai^2 + (1 - a)i + 5$

$= 3(-i) + 2a + (1 - a)i + 5$ $\quad [\because i^3 = -i \text{ and } i^2 = -1]$

$= (2a + 5) + i(1 - a - 3)$, which will be real,

if $1 - a - 3 = 0$,

i.e. $a = -2$.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 Which of the following options define 'imaginary number'?
(a) Square root of any number
(b) Square root of positive number
(c) Square root of negative number
(d) Cube root of number

2 Two complex numbers are equal, if and only if
(a) their real and imaginary parts are separately equal
(b) their real parts are only equal
(c) their imaginary parts are only equal
(d) None of the above

3. If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then the values of x and y are
(a) $x = 3, y = 4$ (b) $x = \frac{3}{4}, y = \frac{33}{4}$
(c) $x = 4, y = 3$ (d) $x = 33, y = 4$

4 Which of the following represent correct form of set of complex numbers?
(a) $C = \{x + iy : x \in R, y \in R \text{ and } i = \sqrt{-1}\}$
(b) $C = \{x + iy : x \in R, y \in I\}$
(c) $C = \{x + iy : x \in I, y \in I\}$ (d) All of these

5 If $x, y \in R$, then $x + iy$ is a non-real complex number, if **[NCERT Exemplar]**
(a) $x = 0$ (b) $y = 0$ (c) $x \neq 0$ (d) $y \neq 0$

VERY SHORT ANSWER Type Questions

6 Write the following as complex numbers.
(i) $\sqrt{-27}$ (ii) $\sqrt{-16}$
(iii) $4 - \sqrt{-5}$ (iv) $-1 - 1\sqrt{-5}$
(v) $1 + \sqrt{-1}$

7 Write the real and imaginary parts of the complex number.
(i) $z = \dfrac{\sqrt{17}}{2} + \dfrac{2}{\sqrt{70}}i$
(ii) $\sqrt{37} + \sqrt{-19}$

8 Write the real and imaginary parts of the following complex numbers.
(i) $2 - i\sqrt{2}$ (ii) $-\dfrac{1}{5} + \dfrac{i}{5}$
(iii) $\dfrac{\sqrt{5}}{7}i$ (iv) $\sqrt{37} + \sqrt{-19}$
(v) $\sqrt{\dfrac{37}{3}} + \dfrac{3}{\sqrt{70}}i$

9 Find a and b such that $2a + 4bi$ and $2i$ represent the same complex number.

10 Find the values of x and y, if

$$x + i(3x - y) = 3 - 6i.$$

11 Write the following as complex numbers.

(i) $5 - 7\sqrt{-21}$ (ii) $\sqrt{x}; x > 0$

(iii) $\frac{\sqrt{3}}{2} - \frac{\sqrt{-2}}{\sqrt{7}}$ (iv) $-b + \sqrt{-4ac}; a, c > 0$

12 Evaluate $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$.

13 Express the following in the form of $a + ib$.

(i) i^{-35} (ii) i^{998}

(iii) $(\sqrt{-1})^{90}$ (iv) $\left(i^{37} \times \frac{1}{i^{67}}\right)$

14 Evaluate $\left[i^{29} + \left(\frac{1}{i}\right)^{50}\right]$.

15 Evaluate the following

(i) i^{80} (ii) $\frac{1}{i}$

(iii) $(-\sqrt{-1})^{31}$ (iv) $\frac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3}$

SHORT ANSWER Type Questions

16 Simplify the following.

(i) $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25}$

(ii) $1 + i^{10} + i^{110} + i^{1000}$

(iii) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

(iv) $\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2$

(v) $(-i)^{4n+3}$, where n is a positive integer.

(vi) $(2i)^3$ (vii) i^{-35}

(viii) i^{-39} (ix) $i^9 + i^{19}$

(x) $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$ (xi) $i^6 + i^8$

(xii) $i + i^2 + i^3 + i^4$ (xiii) $i^{12} + i^{13} + i^{14} + i^{15}$

(xiv) $i^4 + i^8 + i^{12} + i^{16}$

17 Prove that $i^{107} + i^{112} + i^{117} + i^{122} = 0$.

18 Simplify $i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$.

19 Explain the fallacy in the following

$$-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1}$$
$$= \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

HINTS & ANSWERS

1. (c)

2. (a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal, if $a = c$ and $b = d$ i.e., if their real and imaginary parts are separately equal.

3. (b) We have, $4x + i(3x - y) = 3 + i(-6)$ ...(i)

Equating the real and the imaginary parts of Eq. (i), we get $4x = 3, 3x - y = -6$

which on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

4. (a) Set of complex numbers can be represented as $C = \{x + iy : x, y \in R \text{ and } i = \sqrt{-1}\}$

5. (d) Given that, $x, y \in R$

Then, $x + iy$ is non-real complex number if and only if $y \neq 0$.

6. (i) $\sqrt{-27} = 3\sqrt{3}\sqrt{-1}$ **Ans.** $0 + i3\sqrt{3}$

(ii) $0 + 4i$

(iii) $4 - \sqrt{-5} = 4 - \sqrt{5}\sqrt{-1}$ **Ans.** $4 - i\sqrt{5}$

(iv) $-1 - i\sqrt{5}$ (v) $1 + i$

7. (i) Let $z = \sqrt{37} + \sqrt{-19}$

Then, $z = \sqrt{37} + \sqrt{19(-1)} = \sqrt{37} + i\sqrt{19}$

Ans. Re $(z) = 37$ and Im $(z) = \sqrt{19}$

(ii) $\text{Re}(z) = \frac{\sqrt{17}}{2}$, $\text{Im}(z) = \frac{2}{\sqrt{70}}$

8. (i) $2; -\sqrt{2}$ (ii) $-\frac{1}{5}; \frac{1}{5}$ (iii) $0, \frac{\sqrt{5}}{7}$

(iv) $\sqrt{37}; \sqrt{19}$ (v) $\sqrt{\frac{37}{3}}; \frac{3}{\sqrt{70}}$

9. Given, $2a + i4b = 0 + i2$

$\therefore$ $2a = 0$ and $4b = 2$ **Ans.** $a = 0$ and $b = \frac{1}{2}$

10. Solve as Example 2. **Ans.** $x = 3$ and $y = 15$

11. (i) $5 - 7\sqrt{-21} = 5 - 7\sqrt{21(-1)}$

Ans. $5 - 7\sqrt{21}i$

(ii) $\sqrt{x} + 0i$ (iii) $\frac{\sqrt{3}}{2} - i\frac{\sqrt{2}}{7}$

(iv) $-b + \sqrt{-4ac} = -b + \sqrt{(4ac)(-1)}$

Ans. $-b + i2\sqrt{ac}$

12. Solve as Example 3. **Ans.** 0

13. (i) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{i^{4 \times 8 + 3}}$ **Ans.** $(0 + i)$

(ii) $i^{998} = i^{4 \times 249 + 2}$ **Ans.** $-1 + 0i$

(iii) $-1 + 0i$ (iv) $-1 + 0i$

14. $i - 1$

15. (i) $i^{80} = (i^4)^{20} = 1^{20}$ **Ans.** 1

(ii) $-i$ (iii) i

(iv) $\dfrac{i^2+i^4+i^6+i^7}{1+i^2+i^3} = \dfrac{(-1)+(1)+(-1)+(-i)}{1+(-1)+(-i)} = \dfrac{-1-i}{-i} = \dfrac{1}{i}+1$

Ans. $1-i$

16. (i) $2i^2+6i^3+3i^{16}-6i^{19}+4i^{25}$
$= 2(-1)+6(-i)+3(1)-6(-i)+4(i)$

Ans. $1+4i$

(ii) 0 (iii) 0 (iv) $2i$ (v) i
(vi) $8i$ (vii) i (viii) i (ix) 0
(x) $2-2i$ (xi) 0 (xii) 0 (xiii) 0
(xiv) 4

17. $i^{4\times 26+3} + i^{4\times 28+0} + i^{4\times 29+1} + i^{4\times 30+2}$
$= (i^4)^{26}\cdot i^3 + (i^4)^{28}\cdot i^0 + (i^4)^{29}\cdot i + (i^4)^{30}\cdot i^2$
$= (1)^{26}(-i) + (1)^{28}\cdot 1 + (1)^{29}\cdot i + (1)^{30}\cdot(-1) = 0$

18. Given expression $= i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46})$
$= i^n(i^{100}+i^{50}+i^{48}+i^{46}) = i^n(1-1+1-1) = i^n\cdot 0$

Ans. 0

19. Given, $-1 = i.i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

Here, we have $\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)}$

This is not correct as $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ if and only if atleast one of a and b is non-negative. Infact,
$\sqrt{-1}\sqrt{-1} = i\cdot i = i^2 = -1$

|TOPIC 2|

Algebra of Complex Numbers

In this section, we shall study how to add, subtract, multiply and divide the complex numbers.

Addition of Two Complex Numbers

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers, then their addition is defined as

$$z = z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$$

e.g. (i) $(5+3i)+(-4-i) = (5-4)+i(3-1) = 1+2i$

(ii) $(2+3i)+(-6+7i) = (2-6)+i(3+7) = -4+10i$

Note

It can be observed that

(i) Real part of $(z_1+z_2) = \text{Re}(z_1+z_2) = \text{Re}(z_1)+\text{Re}(z_2)$

(ii) Imaginary part of $(z_1+z_2) = \text{Im}(z_1+z_2) = \text{Im}(z_1)+\text{Im}(z_2)$

PROPERTIES OF ADDITION OF COMPLEX NUMBERS

The addition of complex numbers satisfy the following properties

(i) **Closure Law** If z_1 and z_2 are any two complex numbers, then z_1+z_2 is also a complex number.

(ii) **Commutative Law** If z_1 and z_2 are two complex numbers, then $z_1+z_2 = z_2+z_1$.

(iii) **Associative Law** If z_1, z_2 and z_3 are any three complex numbers, then

$$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$$

(iv) **Existence of Additive Identity** There exists the complex number $0 = 0 + 0\,i$ called the identity element for addition (or simply additive identity) i.e. $z+0 = z = 0+z$ for all $z \in C$.

(v) **Existence of Additive Inverse** For every complex number $z = a+ib$, there exists $-z = (-a)+i(-b)$ such that $z+(-z) = 0 = (-z)+z$.

Here, complex number $(-z)$, is called the additive inverse of z.

e.g. Additive inverse of $z = (-4+3i)$ is

$$-z = -(-4+3i) = (4-3i)$$

Subtraction of Two Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, their subtraction $z_1 - z_2$ is defined as the addition of z_1 and $(-z_2)$.

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1+ib_1)+(-a_2-ib_2)$

$$z_1 - z_2 = (a_1-a_2) + i(b_1-b_2)$$

e.g. (i) $(-4+7i)-(-11-23i) = (-4+7i)+(11+23i)$
$= (-4+11)+(7+23)i = 7+30i$

(ii) $(6+5i)-(3+2i) = (6+5i)+(-3-2i)$
$= (6-3)+(5-2)\,i = 3+3i$

Note

It can be observed that

(i) Real part of $(z_1-z_2) = \text{Re}(z_1-z_2) = \text{Re}(z_1)-\text{Re}(z_2)$

(ii) Imaginary part of $(z_1-z_2) = \text{Im}(z_1-z_2) = \text{Im}(z_1)-\text{Im}(z_2)$

EXAMPLE |1| Express the following in the form of $a+ib$.

(i) $\left[\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right]-\left(-\frac{4}{3}+i\right)$ **[NCERT]**

(ii) $\left(\frac{1}{2}+\frac{5}{2}i\right)-\frac{3}{2}i+\left(-\frac{5}{2}-i\right)$

Sol. (i) Consider the given expression,

$$\left[\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right]-\left(-\frac{4}{3}+i\right)$$

$$=\left[\left(\frac{1}{3}+4\right)+i\left(\frac{7}{3}+\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$$

$$=\left(\frac{13}{3}+\frac{8}{3}i\right)+\left(\frac{4}{3}-i\right)=\left(\frac{13}{3}+\frac{4}{3}\right)+i\left(\frac{8}{3}-1\right)$$

$=\frac{17}{3}+\frac{5}{3}i$, which is in the form of $a+ib$.

(ii) $\left(\frac{1}{2}+\frac{5}{2}i\right)-\frac{3}{2}i+\left(-\frac{5}{2}-i\right)=\left(\frac{1}{2}-\frac{5}{2}\right)+i\left(\frac{5}{2}-\frac{3}{2}-1\right)$

$=-2+i0$, which is in the form of $a+ib$.

Note

For expressing the given expression in standard form i.e. in the form of $a+ib$, just simplify the expression according to the rules of algebra.

EXAMPLE |2| Find the real values of x and y, if

$$(x^4+2xi)-(3x^2+iy)=(3-5i)+(1+2iy).$$

(i) Firstly, separate real and imaginary parts of both sides.

(ii) Second, equate the real and imaginary parts of both sides and get equations in terms of x and y.

(iii) Further, solve these equations to get the values of x and y.

Sol. We have, $(x^4+2xi)-(3x^2+iy)=(3-5i)+(1+2iy)$

$\Rightarrow \quad (x^4-3x^2)+(2x-y)i=4+(-5+2y)i$

On equating real and imaginary parts both sides, we get

$$x^4-3x^2=4 \quad \text{... (i)}$$

and $\quad 2x-y=-5+2y \Rightarrow 2x-3y=-5 \quad \text{...(ii)}$

On solving Eq. (i), we get

$$x^4-3x^2=4 \Rightarrow x^4-3x^2-4=0$$

$\Rightarrow \quad x^4-4x^2+x^2-4=0$

$\Rightarrow (x^2-4)(x^2+1)=0 \Rightarrow x^2-4=0$

[$\because x^2+1\neq 0$, for any real value of x]

$\therefore \quad x=\pm 2$

On putting $x=\pm 2$ in Eq. (ii), we get

$y=3$, when $x=2$ and $y=\frac{1}{3}$, when $x=-2$

Thus, $x=-2$, $y=\frac{1}{3}$ or $x=2$, $y=3$.

Multiplication of Two Complex Numbers

The product of two complex numbers $z_1=a+ib$ and $z_2=c+id$ can be as follow

$$z_1z_2=(a+ib)(c+id)=ac+iad+ibc+i^2bd$$

$$=ac+i(ad+bc)+(-1)bd \quad [\because i^2=-1]$$

$$z_1z_2=(ac-bd)+i(ad+bc)$$

e.g. (i) $(2+9i)(11+3i)$

$=2\times 11+2\times 3i+11\times 9i+9\times 3i^2$

$=22+6i+99i-27=-5+105i \quad [\because i^2=-1]$

(ii) $(-5+7i)(-13-3i)$

$=(-5)(-13)+(-5)(-3i)+(7i)(-13)+(7i)(-3i)$

$=65+15i-91i-21i^2=65-76i+21 \quad [\because i^2=-1]$

$=86-76i$

(iii) $(5i)\left(\frac{-3}{5}i\right)=\left[5\times\left(\frac{-3}{5}\right)\right](i\times i)$

$=(-3)(i^2)=-3\times -1=3$

PROPERTIES OF MULTIPLICATION OF COMPLEX NUMBERS

(i) **Closure Law** If z_1 and z_2 are any two complex numbers, then $z_1 z_2$ is also a complex number.

(ii) **Commutative Law** If z_1 and z_2 are any two complex numbers, then $z_1z_2=z_2z_1$.

(iii) **Associative Law** If z_1, z_2 and z_3 are any three complex numbers, then $(z_1z_2)z_3=z_1(z_2z_3)$.

(iv) **Existence of Multiplicative Identity** There exists the complex number $1=1+0\cdot i$ is the **identity** element for multiplication i.e. for every complex number z, we have $z\cdot 1=1\cdot z=z$.

(v) **Existence of Multiplicative Inverse** (or Reciprocal) Corresponding to every non-zero complex number $z=a+ib$, there exists a complex number $z_1=x+iy$ such that $z\cdot z_1=1=z_1\cdot z$, where

$$x=\frac{a}{a^2+b^2} \text{ and } y=\frac{-b}{a^2+b^2}.$$

Then, z_1 is called multiplicative inverse of z and it is denoted by $\frac{1}{z}$ or z^{-1}. We also called z_1, the **reciprocal** of z.

e.g. Let $z=3-7i$. Then, $a=3, b=-7$

Its multiplicative inverse,

$$z^{-1}=\left[\frac{3}{(3)^2+(-7)^2}\right]+i\left[\frac{-(-7)}{(3)^2+(-7)^2}\right]$$

$$=\left(\frac{3}{9+49}\right)+i\left(\frac{7}{9+49}\right)=\frac{3}{58}+\frac{7i}{58}$$

(vi) **Distributive Law** If z_1, z_2 and z_3 are any three complex numbers.

Then, $z_1(z_2+z_3)=z_1z_2+z_1z_3$ [left distributive law]

and $\quad (z_1+z_2)z_3=z_1z_3+z_2z_3$

[right distributive law]

EXAMPLE |3| If z_1 and z_2 are complex numbers, then prove that $\text{Re}(z_1 z_2) = \text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)$. [NCERT]

Sol. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then, $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$\therefore$ $\text{Re}(z_1z_2) = x_1x_2 - y_1y_2$

$= \text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\text{Im}(z_2)$

Hence proved.

EXAMPLE |4| Express the following in the form $a + ib$

(i) $(-i)(3i)\left(-\frac{1}{6}i\right)^3$ (ii) $(-\sqrt{3} + \sqrt{-2})(-2 + \sqrt{-3})$

Sol. (i) $(-i)(3i)\left(-\frac{1}{6}i\right)^3 = (-3i^2)\left(-\frac{1}{216}i^3\right)$

$= (-3 \times (-1))\left(-\frac{1}{216}(-i)\right)$ $[\because i^2 = -1 \text{ and } i^3 = -i]$

$= 3 \times \frac{1}{216} \times i = \frac{i}{72} = 0 + \frac{1}{72}i$

which is in the form of $a + ib$.

(ii) $(-\sqrt{3} + \sqrt{-2})(-2 + \sqrt{-3})$

$= (-\sqrt{3} + i\sqrt{2})(-2 + i\sqrt{3})$

$[\because \sqrt{-2} = \sqrt{2} \times \sqrt{-1} = \sqrt{2}\,i, \text{ similarly } \sqrt{-3} = \sqrt{3}\,i]$

$= 2\sqrt{3} - 3i - 2\sqrt{2}\,i + i^2\sqrt{6}$

$= 2\sqrt{3} - i(3 + 2\sqrt{2}) - \sqrt{6}$ $[\because i^2 = -1]$

$= (2\sqrt{3} - \sqrt{6}) - i(3 + 2\sqrt{2})$

which is in the form of $a + ib$.

EXAMPLE |5| Find the real values of x and y, if $(1 + i)(x + iy) = 2 - 5i$.

Sol. We have, $(1 + i)(x + iy) = 2 - 5i$

$\Rightarrow$ $x + iy + ix + i^2y = 2 - 5i$

$\Rightarrow$ $x + i(y + x) - y = 2 - 5i$ $[\because i^2 = -1]$

$\Rightarrow$ $(x - y) + i(x + y) = 2 - 5i$

On equating real and imaginary parts from both sides, we get

$x - y = 2$ ...(i)

and $x + y = -5$ ...(ii)

On adding Eqs. (i) and (ii), we get

$x - y + x + y = 2 - 5 \Rightarrow 2x = -3 \Rightarrow x = \frac{-3}{2}$

On substituting $x = \frac{-3}{2}$ in Eq. (ii), we get

$\frac{-3}{2} + y = -5 \Rightarrow y = -5 + \frac{3}{2} = \frac{-10 + 3}{2} = \frac{-7}{2}$

$\therefore$ $x = \frac{-3}{2}$ and $y = \frac{-7}{2}$

IDENTITIES RELATED TO COMPLEX NUMBERS

Identity is an equation which is true for all values of the variable (complex number) involved in it. Here, we have the following identities.

(i) $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$, for all complex numbers z_1 and z_2.

(ii) $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$

(iii) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(iv) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

(v) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

Proof (i) We have, $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$

$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2$

[assume first bracket as one term and then apply distributive law]

$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ [by distributive law]

$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2$

[by commutative law of multiplication]

$= z_1^2 + 2z_1z_2 + z_2^2$

Similarly, we can prove the other identities.

Note

Many other identities which are true for all real numbers, can be true for all complex numbers.

EXAMPLE |6| Simplify each of the following and put it in the form $a + ib$.

(i) $(2 + \sqrt{-3})^2$ (ii) $\left(\frac{1}{3} + 3i\right)^3$ [NCERT]

(iii) $(3 + \sqrt{-5})(3 - \sqrt{-5})$

Sol. (i) $(2 + \sqrt{-3})^2 = (2 + \sqrt{3}i)^2$

$= 2^2 + 2\cdot(2)(\sqrt{3}i) + (\sqrt{3}i)^2$

$[\because (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2]$

$= 4 + 4\sqrt{3}i + 3i^2 = 4 + 4\sqrt{3}i - 3$

$= 1 + 4\sqrt{3}\,i$ $[\because i = -1]$

(ii) $\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^2(3i) + 3\left(\frac{1}{3}\right)(3i)^2 + (3i)^3$

$[\because (z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3]$

$= \frac{1}{27} + 3\left(\frac{1}{9}\right)(3i) + 3\left(\frac{1}{3}\right)(9i^2) + 27i^3$

$= \frac{1}{27} + i + 9(-1) + 27(-i)$ $[\because i^2 = -1 \text{ and } i^3 = -i]$

$= \frac{1}{27} - 9 - 26i = \frac{1 - 243}{27} - 26i = \frac{-242}{27} - 26i$

(iii) $(3+\sqrt{-5})(3-\sqrt{-5}) = (3+\sqrt{5}i)(3-\sqrt{5}i)$

$= (3)^2 - (\sqrt{5}i)^2$ $\quad [\because (z_1 - z_2)(z_1 + z_2) = z_1^2 - z_2^2]$

$= 9 - 5i^2 = 9 + 5 = 14$ $\quad [\because i^2 = -1]$

EXAMPLE |7| Express $(1-i)^4$ in the form $a+ib$. **[NCERT]**

Sol. $(1-i)^4 = ((1-i)^2)^2 = ((1)^2 - 2(1)(i) + (i)^2)^2$

$[\because (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2]$

$= (1 - 2i - 1)^2$ $\quad [\because i^2 = -1]$

$= (-2i)^2 = 4i^2 = -4 = -4 + 0i$

which is in the form of $a+ib$.

EXAMPLE |8| Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$. **[NCERT]**

Sol. $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = \left[i^{4\times 4+2} + \left(\frac{i}{i^2}\right)^{25}\right]^3$

$= \left[(i^4)^4 \cdot i^2 + \left(\frac{i}{-1}\right)^{25}\right]^3$ $\quad [\because i^2 = -1]$

$= \left[1\cdot(-1) + \frac{i^{25}}{(-1)}\right]^3$ $\quad [\because i^4 = 1 \text{ and } i^2 = -1]$

$= [-1 - i^{4\times 6+1}]^3$

$= [-1 - (i^4)^6 \cdot i]^3 = [-1 - i]^3$ $\quad [\because i^4 = 1]$

$= -[1+i]^3 = -(1 + 3i + 3i^2 + i^3)$

$[\because (z_1 + z_2)^3 = (z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3]$

$= -(1 + 3i - 3 - i)$ $\quad [\because i^2 = -1 \text{ and } i^3 = -i]$

$= -(-2 + 2i) = 2 - 2i = 2(1-i)$

EXAMPLE |9| Evaluate $(1+i)^6 + (1-i)^3$. **[NCERT Exemplar]**

Sol. We have, $(1+i)^6 = ((1+i)^2)^3$

$= (1 + i^2 + 2i)^3$

$[\because (z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2]$

$= (1 - 1 + 2i)^3$ $\quad [\because i^2 = -1]$

$\Rightarrow (1+i)^6 = (2i)^3 = 8i^3 = -8i$ $\quad [\because i^3 = -1]$ ...(i)

and $(1-i)^3 = 1^3 - i^3 - 3(1)^2 i + 3(1)(i)^2$

$[\because (z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3]$

$= 1 - (-i) - 3i - 3$ $\quad [\because i^3 = -i \text{ and } i^2 = -1]$

$\Rightarrow (1-i)^3 = -2 - 2i$ ...(ii)

On adding Eqs. (i) and (ii), we get

$(1+i)^6 + (1-i)^3 = -8i - 2 - 2i = -2 - 10i$

EXAMPLE |10| Find the values of x and y, if $(3x - 2iy)(2+i)^2 = 10(1+i)$.

Sol. We have, $(3x - 2iy)(2+i)^2 = 10(1+i)$

$\Rightarrow (3x - 2iy)(4 + i^2 + 4i) = 10 + 10i$

$[\because (z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2]$

$\Rightarrow (3x - 2iy)(4 - 1 + 4i) = 10 + 10i$ $\quad [\because i^2 = -1]$

$\Rightarrow (3x - 2iy)(3 + 4i) = 10 + 10i$

$\Rightarrow (9x + 8y) + i(12x - 6y) = 10 + 10i$

On equating real and imaginary parts of both sides, we get

$9x + 8y = 10$ ... (i)

and $12x - 6y = 10$ ... (ii)

On multiplying Eq. (i) by 6 and Eq. (ii) by 8, then adding the result, we get

$54x + 48y + 96x - 48y = 60 + 80$

$\Rightarrow 150x = 140 \Rightarrow x = \frac{14}{15}$

On substituting $x = \frac{14}{15}$ in Eq. (i), we get

$9 \times \frac{14}{15} + 8y = 10 \Rightarrow \frac{42}{5} + 8y = 10$

$\Rightarrow 8y = 10 - \frac{42}{5} \Rightarrow 8y = \frac{8}{5} \Rightarrow y = \frac{1}{5}$

$\therefore x = \frac{14}{15} \text{ and } y = \frac{1}{5}$

EXAMPLE |11| If $(x+iy)^{1/3} = a + ib$, where $x, y, a, b \in R$, then show that $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$. **[NCERT Exemplar]**

Firstly, use identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and then equate the coefficients of real and imaginary parts.

Sol. We have, $(x+iy)^{1/3} = a + ib$

$\Rightarrow x + iy = (a + ib)^3$ $\quad$ [cubing both sides]

$\Rightarrow x + iy = a^3 + i^3b^3 + 3a^2bi + 3a\,b^2i^2$

$[\because (z_1 + z_2)^3 = z_1^3 + z_2^3 + 3z_1^2z_2 + 3z_1z_2^2]$

$\Rightarrow x + iy = a^3 - ib^3 + i3a^2b - 3ab^2$

$[\because i^3 = -i \text{ and } i^2 = -1]$

$\Rightarrow x + iy = a^3 - 3ab^2 + i(3a^2b - b^3)$

On equating real and imaginary parts from both sides, we get

$x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3$

$\Rightarrow \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$

Now, $\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2$

$= -2a^2 - 2b^2 = -2(a^2 + b^2)$

EXAMPLE |12| Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}i$.

Sol. We have, $x = -2 - \sqrt{3}i$ **[NCERT Exemplar]**

$\Rightarrow \quad x + 2 = -\sqrt{3}i$

On squaring both sides, we get

$$(x+2)^2 = (-\sqrt{3}i)^2 \Rightarrow x^2 + 4 + 4x = 3i^2$$

$[\because (z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2]$

$\Rightarrow \quad x^2 + 4x + 4 = -3 \qquad [\because i^2 = -1]$

$\Rightarrow \quad x^2 + 4x + 7 = 0$

Now divide $2x^4 + 5x^3 + 7x^2 - x + 41$ by $x^2 + 4x + 7$.

$$\begin{array}{r|l} & 2x^2 - 3x + 5 \\ x^2 + 4x + 7 & 2x^4 + 5x^3 + 7x^2 - x + 41 \\ & 2x^4 + 8x^3 + 14x^2 \\ & - \quad - \quad - \\ \hline & -3x^3 - 7x^2 - x + 41 \\ & -3x^3 - 12x^2 - 21x \\ & + \quad + \quad + \\ \hline & 5x^2 + 20x + 41 \\ & 5x^2 + 20x + 35 \\ & - \quad - \quad - \\ \hline & 6 \end{array}$$

Thus, $2x^4 + 5x^3 + 7x^2 - x + 41$

$= (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$[\because \text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}]$

$= 0 \times (2x^2 - 3x + 5) + 6 = 6 \quad [\because x^2 + 4x + 7 = 0]$

Division of Two Complex Numbers

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Therefore, $\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = z_1 \cdot \left(\frac{1}{z_2}\right)$

Note

Order relations "greater than" and "less than" are not defined for complex numbers.

METHOD FOR EXPRESSING DIVISION OF COMPLEX NUMBERS IN THE STANDARD FORM

***Step* I** Simplify the numerator and denominator separately and convert it in the form of $\frac{a+ib}{c+id}$.

***Step* II** On rationalising the denominator of the result obtained in step I, i.e. multiply the numerator and denominator by $c - id$.

***Step* III** Simplify and write it in the $x + iy$ form.

EXAMPLE |13| Express $(-2-5i) \div (3-6i)$ in the form $a + ib$.

Sol. $(-2-5i) \div (3-6i) = \frac{-2-5i}{3-6i}$

$= \frac{-(2+5i)}{3-6i} \times \frac{(3+6i)}{(3+6i)}$

[by rationalising the denominator]

$= -\frac{[6 + 12i + 15i + 30i^2]}{(3)^2 - (6i)^2} \quad [\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2]$

$= \frac{-[6 + 27i - 30]}{9 + 36} \quad [\because i^2 = -1]$

$= \frac{-(-24 + 27i)}{45} = \frac{24}{45} - \frac{27}{45}i$

$= \frac{8}{15} - \frac{3}{5}i = \frac{8}{5} + i\left(\frac{-3}{5}\right)$, which is in the form of $(a + ib)$.

EXAMPLE |14| Express $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$ in the form of $a + ib$. **[NCERT]**

Write the complex number in the form $\frac{a+ib}{c+id}$ and then rationalising the denominator. Further simplify it.

Sol. $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

$= \frac{(3)^2 - (\sqrt{5}i)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \quad [\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2]$

$= \frac{9+5}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i}$

[by rationalising the denominator]

$= \frac{7\sqrt{2}i}{2i^2} = \frac{7\sqrt{2}i}{-2} = 0 - i\frac{7\sqrt{2}}{2}$

$= 0 + i\left(\frac{-7\sqrt{2}}{2}\right)$, which is in the form of $(a + ib)$.

EXAMPLE |15| Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form. **[NCERT]**

Sol. $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \left[\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\right]\left(\frac{3-4i}{5+i}\right)$

$= \left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)$

$= \left(\frac{-1+9i}{1-3i+4}\right)\left(\frac{3-4i}{5+i}\right) \quad [\because i^2 = -1]$

$$=\left(\frac{-1+9i}{5-3i}\right)\left(\frac{3-4i}{5+i}\right)=\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}$$

$$=\frac{-3+31i+36}{25-10i+3}=\frac{33+31i}{28-10i} \quad [\because i^2=-1]$$

$$=\frac{(33+31i)}{(28-10i)}\times\frac{(28+10i)}{(28+10i)}$$

[by rationalising the denominator]

$$=\frac{924+868i+330i+310i^2}{784-100i^2}$$

$$=\frac{924+1198i-310}{784+100}$$

$$=\frac{614+1198i}{884}=\frac{307}{442}+\frac{599}{442}i$$

EXAMPLE |16| Express $\left(\frac{4i^3-1}{2i+1}\right)^2$ in the form of $a+ib$, where $a, b \in R$.

Sol. $\left(\frac{4i^3-1}{2i+1}\right)^2=\left[\frac{4\cdot(-i)-1}{2i+1}\right]^2 \quad [\because i^3=-i]$

$$=\left(\frac{-4i-1}{2i+1}\right)^2=\frac{(1+4i)^2}{(1+2i)^2}=\frac{1+16i^2+8i}{1+4i^2+4i}$$

$[\because (z_1+z_2)^2=z_1^2+z_2^2+2z_1z_2]$

$$=\frac{1-16+8i}{1-4+4i}=\frac{-15+8i}{-3+4i} \quad [\because i^2=-1]$$

$$=\frac{15-8i}{3-4i}=\frac{15-8i}{3-4i}\times\frac{3+4i}{3+4i}$$

[by rationalising the denominator]

$$=\frac{(15-8i)(3+4i)}{9^2-16i^2} \quad [\because (z_1-z_2)(z_1+z_2)=z_1^2-z_2^2]$$

$$=\frac{(15-8i)(3+4i)}{9-16i^2}=\frac{45+60i-24i-32i^2}{9+16}$$

$$=\frac{45+36i+32}{25}=\frac{77+36i}{25}=\frac{77}{25}+\frac{36}{25}i$$

which is in the form of $(a+ib)$.

EXAMPLE |17| If $a+ib=\frac{x+i}{x-i}$, where x is real, then prove that $a^2+b^2=1$ and $\frac{b}{a}=\frac{2x}{x^2-1}$. **[NCERT]**

Sol. We have, $a+ib=\frac{x+i}{x-i}=\frac{x+i}{x-i}\times\frac{x+i}{x+i}$

[by rationalising the denominator]

$$=\frac{x^2+2xi+i^2}{x^2-i^2}=\frac{x^2-1+2xi}{x^2+1} \quad [\because i^2=-1]$$

$$\Rightarrow \quad a+ib=\frac{x^2-1}{x^2+1}+\frac{2x}{x^2+1}i$$

On comparing real and imaginary parts both sides, we get

$$a=\frac{x^2-1}{x^2+1} \text{ and } b=\frac{2x}{x^2+1} \quad \text{...(i)}$$

Now, $a^2+b^2=\left(\frac{x^2-1}{x^2+1}\right)^2+\left(\frac{2x}{x^2+1}\right)^2$ [from Eq. (i)]

$$=\frac{(x^2-1)^2+4x^2}{(x^2+1)^2}=\frac{x^4+1-2x^2+4x^2}{(x^2+1)^2}$$

$$=\frac{x^4+1+2x^2}{(x^2+1)^2}=\frac{(x^2+1)^2}{(x^2+1)^2}=1$$

Also, $\frac{b}{a}=\frac{\frac{2x}{x^2+1}}{\frac{x^2-1}{x^2+1}}=\frac{2x}{x^2-1}$ [from Eq. (i)]

Hence proved.

EXAMPLE |18| If $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$, then find (x, y). **[NCERT Exemplar]**

Sol. Consider, $\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}$

[by rationalising the denominator]

$$=\frac{(1+i)^2}{1-i^2}=\frac{1+i^2+2i}{1+1}$$

$$\Rightarrow \quad \frac{1+i}{1-i}=\frac{1-1+2i}{2}=i \quad [\because i^2=-1] \text{...(i)}$$

Now, $\frac{1-i}{1+i}=\frac{1}{\left(\frac{1+i}{1-i}\right)}=\frac{1}{i}$ [from Eq. (i)]

$$=\frac{1}{i}\times\frac{i}{i}=\frac{i}{i^2}=\frac{i}{(-1)}=-i \quad [\because i^2=-1] \text{...(ii)}$$

Hence, $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=i^3-(-i)^3$

$$=i^3+i^3=2i^3=2(-i)=0-2i \quad [\because i^3=-i]$$

$\therefore \quad x+iy=0-2i$

On comparing real and imaginary parts both sides, we get

$$x=0 \text{ and } y=-2$$

$\therefore \quad (x, y)=(0, -2)$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 If z is a complex number and $z+(-z)=0$, then

(a) $(-z)$ is called additive inverse of z
(b) $-z$ is additive identity of z
(c) $-z$ is closure of z
(d) $-z$ is commutative of z

2 If z is non-zero complex number and $z=a+ib$, then inverse of z is

(a) $\frac{a}{a^2+b^2}+\frac{-bi}{a^2+b^2}$ (b) $\frac{a}{a^2-b^2}+\frac{-bi}{a^2-b^2}$
(c) $\frac{a}{a^2-b^2}+\frac{ib}{a^2-b^2}$ (d) $\frac{-a}{a^2+b^2}+\frac{-bi}{a^2+b^2}$

3 If $z_1=6+3i$ and $z_2=2-i$, then $\frac{z_1}{z_2}$ is equal to

(a) $\frac{1}{5}(9+12i)$ (b) $9+12i$
(c) $3+2i$ (d) $\frac{1}{5}(12+9i)$

4 If $z=i^{-39}$, then simplest form of z is equal to

(a) $1+0i$ (b) $0+i$ (c) $0+0i$ (d) $1+i$

5 If $(x-iy)^{1/3}=a+ib$, where $x,y,a,b\in R$ then the value of $\frac{x}{a}+\frac{y}{b}$ is equal to

(a) $4(a^2-b^2)$ (b) $4(a^2+b^2)$
(c) $2(a^2-b^2)$ (d) $2(a^2+b^2)$

VERY SHORT ANSWER Type Questions

6 Express the following in the form $a+ib$.

(i) $\left(\frac{1}{5}+\frac{2}{5}i\right)-\left(4+i\frac{5}{2}\right)$ [NCERT]
(ii) $3(1-2i)-(-4-5i)+(-8+3i)$

7 Find the real values of x and y for which $(1+i)y^2+(6+i)=(2+i)x$.

8 Find the sum of the complex numbers $-\sqrt{3}+\sqrt{-2}$ and $2\sqrt{3}-i$.

9 Express $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$ in the form of $a+ib$.

10 Find the real values of x and y for which $(x+iy)(2-3i)=4+i$.

11 Express $(\sqrt{6}+5i)\left(\sqrt{6}-\frac{1}{5}i\right)$ in the form of $a+ib$.

12 Express $(7+5i)(7-5i)$ in the form of $a+ib$.

13 Express the following in the form of $a+ib$.

(i) $(7-i2)-(4+i)+(-3+i5)$
(ii) $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$
(iii) $i^3+(6+3i)-(20+5i)+(14+3i)$
(iv) $(7+i5)(7-i5)$ (v) $3i^3(15i^6)$
(vi) $\sqrt{3}+(\sqrt{3}-i2)-(3-2i)$

SHORT ANSWER Type Questions

14 Express $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ in the form $a+ib$.

15 Evaluate $\frac{(1-i)^3}{1-i^3}$. [NCERT Exemplar]

16 If $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$, then find (a,b).

17 If $\frac{(1+i)^2}{2-i}=x+iy$, then find the value of $x+y$.

18 Express $\left[\left(\sqrt{5}+\frac{i}{2}\right)(\sqrt{5}-2i)\right]\div(6+5i)$ in the form $a+ib$.

19 If $x+iy=\frac{a+i}{a-i}$, then prove that $ay-1=x$.

20 Express $(5-3i)^3$ in the form $a+ib$. [NCERT]

21 Express the following in the form $a+ib$.

(i) $(1+i)^4$ (ii) $\left(\frac{1}{2}+i2\right)^3$
(iii) $\left(-2-i\frac{1}{3}\right)^3$ (iv) $\left(\frac{1}{3}+3i\right)^3$
(v) $(5-3i)^3$ (vi) $(1-i)^4$

22 What is the smallest positive integer n, for which $(1+i)^{2n}=(1-i)^{2n}$?

23 If $z_1,z_2\in C$, prove that
$\text{Im}(z_1\cdot z_2)=\text{Re}(z_1)\cdot\text{Im}(z_2)+\text{Im}(z_1)\cdot\text{Re}(z_2)$.

24 Find x and y, if $(3x-2iy)(2+i)^2=10(1+i)$.

25 Find the real values of x and y, if $\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$.

26 Find the following as a single complex number $x+iy$.

(i) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}$
(ii) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

LONG ANSWER Type Questions

27 Find the values of x and y, if

$$\frac{(1+i)\,x-2i}{3+i}+\frac{(2-3i)\,y+i}{3-i}=i.$$

28 If $x=\sqrt{-2}-1$, then find the value of $x^4+4x^3+6x^2+4x+9$.

29 If $x=3+4i$, then find the value of $x^4-12x^3+70x^2-204x+225$.

30 If $x=3+2i$, then find the value of $x^4-4x^3+4x^2+8x+39$.

31 If $x=-5+2\sqrt{-4}$, find the value of $x^4+9x^3+35x^2-x+4$.

32 Evaluate $2x^3+2x^2-7x+72$, when $x=\dfrac{3-5i}{2}$.

HINTS & ANSWERS

1. (a)

2. (a) Given, $z=a+ib$. Let multiplicative inverse of z is z^{-1}.

Then, $z^{-1}=\dfrac{1}{z}=\dfrac{1}{a+ib}=\dfrac{a-ib}{(a+ib)(a-ib)}$

[multiplying numerator and denominator by $(a-ib)$]

$$=\frac{a-ib}{a^2+b^2}$$

$$z^{-1}=\frac{a}{a^2+b^2}-i\,\frac{b}{a^2+b^2}$$

3. (a) We have,

$z_1=6+3i$ and $z_2=2-i$

$$\therefore\ \frac{z_1}{z_2}=(6+3i)\,\frac{1}{2-i}=\frac{(6+3i)(2+i)}{(2-i)(2+i)}$$

$$=(6+3i)\left(\frac{2}{5}+i\,\frac{1}{5}\right)$$

$$=(6+3i)\,\frac{(2+i)}{5}$$

$$=\frac{1}{5}(9+12i)$$

4. (b) $i^{-39}=\dfrac{1}{i^{39}}$

Multiplying and dividing by i, we get

$$=\frac{i}{i^{40}}=\frac{i}{(i^4)^{10}}=\frac{i}{(1)^{10}}=\frac{i}{1}=i \qquad (\because i^4=1)$$

$$=0+i$$

5. (a) We have $(x-ij)^{1/3}=a+ib$

$\Rightarrow\quad x-iy=(a+ib)^3$

$\Rightarrow\quad x-iy=a^3+i^3b^3+3abi(a+ib)$

$\Rightarrow\quad x-iy=a^3-b^3i+3a^2bi-3ab^2$

$\Rightarrow\quad x=a^3-3ab^2$ and $y=3a^2b-b^3$

$\Rightarrow\quad \dfrac{x}{a}=a^2-3b^2$ and $\dfrac{y}{b}=3a^2-b^2$

$\therefore\quad \dfrac{x}{a}+\dfrac{y}{b}=a^2-3b^2+3a^2-b^2=4(a^2-b^2)$

6. (i) $-\dfrac{19}{5}-\dfrac{21}{10}i$ (ii) $-1+2i$

7. $y^2+iy^2+6+i=2x+ix \Rightarrow y^2+6=2x$ and $y^2+1=x$

Ans. $(x=5,\ y=2)$ or $(x=5,\ y=-2)$

8. $(-\sqrt{3}+\sqrt{-2})+(2\sqrt{3}-i)=-\sqrt{3}+\sqrt{2}i+2\sqrt{3}-i$

Ans. $\sqrt{3}+(\sqrt{2}-1)i$

9. $(-6+\sqrt{2})+i(\sqrt{3}+2\sqrt{6})$

10. $x=\dfrac{5}{13}$ and $y=\dfrac{14}{13}$

11. $7+\dfrac{24\sqrt{6}}{5}i$

12. Use the identity $(z_1+z_2)(z_1-z_2)=z_1^2-z_2^2$ **Ans.** $74+0i$

13. (i) $0+2i$ (ii) $\dfrac{17}{3}+\dfrac{5}{3}i$ (iii) $0+0i$ (iv) $74+0i$
(v) $0+45i$ (vi) $(2\sqrt{3}-3)+0i$

14. Multiply numerator and denominator by $1+\sqrt{2}i$, we get

$$\frac{5+\sqrt{2}i}{1-\sqrt{2}i}\times\frac{1+\sqrt{2}i}{1+\sqrt{2}i}=\frac{3+6\sqrt{2}i}{1+2}$$ **Ans.** $1+2\sqrt{2}i$

15. $$\frac{(1-i)^3}{1-i^3}=\frac{1^3-i^3-3i+3i^2}{1+i}=\frac{1+i-3i-3}{1+i}=\frac{-2-2i}{1+i}$$

Ans. -2

16. $$\left(\frac{1-i}{1+i}\right)^{100}=\left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)^{100}$$

$$=\left(\frac{1^2+i^2-2i}{1+1}\right)^{100}=\left(\frac{1-1-2i}{2}\right)^{100}=(-i)^{100}=1$$

Ans. (1, 0)

17. $$\frac{(1+i)^2}{2-i}=\frac{1+i^2+2i}{2-i}=\frac{2i}{2-i}\times\frac{2+i}{2+i}=\frac{4i-2}{4+1}$$ **Ans.** $\dfrac{2}{5}$

18. $$\frac{\left(\sqrt{5}+\frac{i}{2}\right)(\sqrt{5}-2i)}{6+5i}=\frac{5+1+\left(\frac{\sqrt{5}}{2}-2\sqrt{5}\right)i}{6+5i}$$

$$=\frac{6-\frac{3\sqrt{5}i}{2}}{(6+5i)}\times\frac{6-5i}{6-5i}=\frac{36-\frac{15\sqrt{5}}{2}+\left(\frac{-18\sqrt{5}}{2}-30\right)i}{36+25}$$

Ans. $\dfrac{72-15\sqrt{5}}{122}-i\left(\dfrac{30+9\sqrt{5}}{61}\right)$

19. $\frac{a+i}{a-i}\times\frac{a+i}{a+i}=\frac{a^2-1+2ai}{a^2+1}$

20. $(5-3i)^3=5^3-(3i)^3-3\times5^2\times3i+3\times5\times(3i)^2$

$=125+27i-225i-135$

Ans. $-10-198i$

21. (i) $(1+i)^4=(1+i)^2(1+i)^2$

$=(1-1+2i)(1-1+2i)=(2i)(2i)$

Ans. $-4+0i$

(ii) $\left(\frac{1}{2}+2i\right)^3=\left(\frac{1}{2}\right)^3+(2i)^3+3\left(\frac{1}{2}\right)^2(2i)+3\left(\frac{1}{2}\right)(2i)^2$

$=\frac{1}{8}-8i+\frac{3i}{2}-6$

Ans. $-\frac{47}{8}-\frac{13}{2}i$

(iii) $-\frac{22}{3}-\frac{107}{27}i$ (iv) $-\frac{242}{27}-26i$

(v) $-10-198i$ (vi) $-4+0i$

22. Write the given expression as $\left(\frac{1+i}{1-i}\right)^{2n}=1$ **Ans.** $n=2$

23. Now, $z_1z_2=(a_1+ib_1)(a_2+ib_2)$

$\Rightarrow\ z_1z_2=a_1a_2-b_1b_2+(b_1a_2+a_1b_2)i$

$\therefore\ \text{Im}(z_1z_2)=a_1b_2+b_1a_2=\text{Re}(z_1)\text{Im}(z_2)+\text{Re}(z_2)\text{Im}(z_1)$

24. Given, $(3x-2iy)(2+i)^2=10(1+i)$

$\Rightarrow\ (3x-2iy)(4+4i+i^2)=10+10i$

$\Rightarrow\ (9x-6yi+12xi-8i^2y)=10+10i$

$\therefore\ 9x+8y=10$ and $12x-6y=10$

Ans. $x=\frac{14}{15},\ y=\frac{1}{5}$

25. $\frac{(x-1)(3-i)+(y-1)(3+i)}{(3+i)(3-i)}=i$

$\Rightarrow\ \frac{(3x+3y-6)+i(y-x)}{9-i^2}=i$ **Ans.** $x=-4,\ y=6$

26. (i) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}=\frac{9+5}{2i\sqrt{2}}=\frac{14}{2\sqrt{2}}\times\frac{i}{i^2}=\frac{-14i}{2\sqrt{2}}$

Ans. $-\frac{7\sqrt{2}}{2}i$

(ii) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$=\frac{(-1+9i)(3-4i)}{(5-3i)(5+i)}=\frac{33+31i}{28-10i}\times\frac{28+10i}{28+10i}$

$=\frac{614+1198i}{784+100}=\frac{614+1198i}{884}$ **Ans.** $\frac{307}{442}+\frac{599}{442}i$

27. Given, $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

$\Rightarrow\ \frac{x+(x-2)i}{3+i}+\frac{2y+(1-3y)i}{3-i}=i$

$\Rightarrow\ \frac{[x+(x-2)i](3-i)+[2y+(1-3y)i](3+i)}{(3+i)(3-i)}=i$

$\Rightarrow\ (4x+9y-3)+i(2x-7y-3)=10i$

$\Rightarrow\ 4x+9y-3=0$ and $2x-7y-3=10.$

Ans. $x=3$ and $y=-1$

28. $x+1=\sqrt{2}i\Rightarrow x^2+1+2x=-2\Rightarrow x^2+2x+3=0$

Further, solve as Example 12. **Ans.** 12

29. $x-3=4i\Rightarrow x^2+9-6x=-16\Rightarrow x^2-6x+25=0$

Further, solve as Example 12. **Ans.** 0

30. 0 **31.** -160 **32.** 4

|TOPIC 3|

Conjugate, Modulus and Argand Plane of Complex Number

Conjugate of a Complex Number

A pair of complex numbers z_1 and z_2 is said to be **conjugate** of each other, if the sum and product of two z_1 and z_2 both are real.

Let $z_1=a+ib$ and $z_2=a-ib$

Sum of z_1 and $z_2=(a+ib)+(a-ib)=2a$ (real)

Product of z_1 and $z_2=(a+ib)(a-ib)$

$=a^2-i^2b^2$ [$\because\ (x+y)(x-y)=x^2-y^2$]

$=a^2+b^2$ (real) [$\because\ i^2=-1$]

Hence, z_1 and z_2 are conjugate to each other.

The conjugate of a complex number z, is the complex number, obtained by changing the sign of imaginary part of z. It is denoted by $\bar{z}$.

e.g. If $z=2+3i$, then $\bar{z}=2-3i$

and if $z=-4-3i$, then $\bar{z}=-4+3i$

Note

(i) A pair of complex numbers z_1 and z_2 is said to be conjugate of each other, if $\bar{z}_1=z_2$ and $\bar{z}_2=z_1$.

(ii) Conjugate of purely real complex number is same. i.e. if $z=3$, then $\bar{z}=3$

EXAMPLE |1| Find the conjugate of complex number $3 + i$.

Sol. Let $z = 3 + i$

$\therefore \bar{z} = 3 - i$

[since, the conjugate of complex number z, is the complex number, obtained by changing the sign of imaginary part of z]

EXAMPLE |2| Simplify the following complex number.

$\overline{9 - i} + \overline{6 + i^3} - \overline{9 + i^2}$

Firstly, write each complex number in standard form and then find its conjugate.

Sol. $\overline{9 - i} + \overline{6 + i^3} - \overline{9 + i^2}$

$= (9 + i) + \overline{6 - i} - \overline{9 - 1}$ [$\because i^3 = -i$ and $i^2 = -1$]

$= (9 + i) + (6 + i) - \bar{8}$

$= 15 + 2i - 8 = 7 + 2i$

EXAMPLE |3| Find the real and imaginary parts of the conjugate of the complex number $-5i^{-15} - 6i^{-8}$.

Firstly, write the given complex number in the form of $a + ib$ and find its conjugate. Further, compare the real and imaginary parts of both sides to get the result.

Sol. Let $z = -5i^{-15} - 6i^{-8}$

$= \frac{-5}{i^{15}} - \frac{6}{i^8} = \frac{-5}{(i^4)^3 \cdot i^3} - \frac{6}{(i^4)^2}$ [$\because i^{15} = i^{4\times 3+3}$]

$= \frac{-5}{(1)^3 \cdot (-i)} - \frac{6}{(1)^2}$ [$\because i^4 = 1$ and $i^3 = -i$]

$= \frac{-5}{-i} - 6 = \frac{5}{i} - 6 = \frac{5 - 6i}{i} = \frac{(5 - 6i)i}{i \cdot i}$

[by rationalising the denominator]

$= \frac{5i - 6i^2}{i^2} = \frac{5i + 6}{-1} = -6 - 5i$ [$\because i^2 = -1$]

$\therefore \bar{z} = -6 + 5i$

Hence, $\text{Re}(\bar{z}) = -6$ and $\text{Im}(\bar{z}) = 5$

EXAMPLE |4| Find the real numbers x and y, if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Firstly, simplify the product of two complex numbers in the form of $a + ib$ and equate it to the conjugate of $-6-24i$ i.e. $-6+24i$. Further, equate real and imaginary parts of both sides and solve the equations to get the values of x and y.

Sol. We have, $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

$\Rightarrow (x - iy)(3 + 5i) = -6 + 24i$

[$\because$ conjugate of $-6 - 24i = -6 + 24i$]

$\Rightarrow 3x - 3iy + 5ix - 5i^2y = -6 + 24i$

$\Rightarrow (3x + 5y) + i(5x - 3y) = -6 + 24i$ [$\because i^2 = -1$] ...(i)

On equating real and imaginary parts both sides of Eq. (i), we get

$3x + 5y = -6$ ...(ii)

and $5x - 3y = 24$ ...(iii)

On multiplying Eq. (i) by 3 and Eq. (ii) by 5, then adding the result, we get

$9x + 15y + 25x - 15y = -18 + 120 \Rightarrow 34x = 102 \Rightarrow x = 3$

On substituting $x = 3$ in Eq. (i), we get

$9 + 5y = -6 \Rightarrow 5y = -15 \Rightarrow y = -3$

Hence, the required values of x and y are respectively 3 and -3.

EXAMPLE |5| Let $z_1 = 2 - i$ and $z_2 = -2 + i$, then find $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$. **[NCERT]**

Sol. We have, $z_1 = 2 - i$ and $z_2 = -2 + i$

Now, $\frac{z_1 z_2}{\bar{z}_1} = \frac{(2 - i)(-2 + i)}{\overline{(2 - i)}} = \frac{-(2 - i)(2 - i)}{2 + i}$

$= -\frac{(4 + i^2 - 4i)}{2 + i} = -\frac{(4 - 1 - 4i)}{2 + i} = -\frac{(3 - 4i)}{2 + i} \times \frac{2 - i}{2 - i}$

[by rationalising the denominator]

$= -\frac{(6 - 3i - 8i + 4i^2)}{4 - i^2}$ [$\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2$]

$= -\frac{(6 - 11i - 4)}{5}$ [$\because i^2 = -1$]

$= -\frac{2 - 11i}{5} = \frac{-2}{5} + \frac{11}{5}i \therefore \text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \text{Re}\left(\frac{-2}{5} + \frac{11i}{5}\right) = \frac{-2}{5}$

EXAMPLE |6| What is the conjugate of $\frac{2 - i}{(1 - 2i)^2}$? **[NCERT Exemplar]**

Firstly, write the given complex number in the standard form and then find its conjugate.

Sol. Let $z = \frac{2 - i}{(1 - 2i)^2} = \frac{2 - i}{(1^2 - 2(1)(2i) + (2i)^2)}$

[$\because (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$]

$\Rightarrow z = \frac{2 - i}{(1 - 4i - 4)}$ [$\because i^2 = -1$]

$\Rightarrow z = \frac{2 - i}{-3 - 4i} \Rightarrow z = \frac{2 - i}{-3 - 4i} \times \frac{-3 + 4i}{-3 + 4i}$

[by rationalising the denominator]

$\Rightarrow z = \frac{(2 - i)(-3 + 4i)}{(-3)^2 - (4i)^2}$

[$\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2$]

$\Rightarrow z = \frac{-6 + 8i + 3i - 4i^2}{9 - 16i^2} \Rightarrow z = \frac{-6 + 11i + 4}{9 + 16}$ [$\because i^2 = -1$]

$\Rightarrow z = \frac{-2 + 11i}{25} \Rightarrow z = -\frac{2}{25} + \frac{11}{25}i$

Hence, $\bar{z} = -\frac{2}{25} - \frac{11}{25}i$

EXAMPLE |7| Solve the equation $z^2 = \bar{z}$, where $z = x + iy$. **[NCERT Exemplar]**

Sol. We have, $z^2 = \bar{z} \Rightarrow (x+iy)^2 = x - iy$

$$\Rightarrow \quad x^2 + (iy)^2 + 2xyi = x - iy \quad [\because (z_1+z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2]$$

$$\Rightarrow \quad x^2 - y^2 + 2xyi = x - iy \quad [\because i^2 = -1]$$

On equating real and imaginary parts, we get

$$x^2 - y^2 = x \quad \text{...(i)}$$

and $$2xy = -y \quad \text{...(ii)}$$

From Eq. (ii), we have

$$2xy + y = 0 \Rightarrow y(2x+1) = 0$$

$$\Rightarrow \quad y = 0 \text{ or } x = -\frac{1}{2}$$

***Case* I** When $y = 0$.

In this case, we have $x^2 = x$ [from Eq. (i)]

$$\Rightarrow \quad x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

$$\therefore \quad z = 0 + 0i \text{ or } z = 1 + 0i$$

***Case* II** When $x = -\frac{1}{2}$.

In this case, we have

$$\frac{1}{4} - y^2 = -\frac{1}{2} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad y^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \quad z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Hence, the solutions of given equation are $0 + 0i, 1 + 0i$,

$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Properties of Conjugate of Complex Numbers

1. $\overline{(\bar{z})} = z$, where $\bar{z}$ is the conjugate of complex number z and $\bar{\bar{z}}$ is the conjugate of complex number $\bar{z}$.
2. $z + \bar{z} = 2\,\text{Re}(z)$
3. $z - \bar{z} = 2i\,\text{Im}(z)$
4. $z = \bar{z} \Leftrightarrow z$ is purely real.
5. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.
6. $z\bar{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$
7. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
8. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
9. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
10. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, provided $z_2, \bar{z}_2 \neq 0$

EXAMPLE |8| If $z_1 = 3 + 2i$ and $z_2 = 2 - i$, then verify that

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

Sol. Given that, $z_1 = 3 + 2i$ and $z_2 = 2 - i$

(i) Now, $z_1 + z_2 = (3+2i) + (2-i) = 5 + i$

$$\Rightarrow \quad \overline{z_1 + z_2} = \overline{5+i} = 5 - i \quad \text{...(i)}$$

Now, consider $\bar{z}_1 + \bar{z}_2 = \overline{(3+2i)} + \overline{(2-i)}$

$$= 3 - 2i + 2 + i = 5 - i \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

(ii) Now, $z_1z_2 = (3+2i)(2-i) = 6 - 3i + 4i - 2i^2$

$$= 6 + i - 2(-1) = 8 + i \quad [\because i^2 = -1]$$

$$\Rightarrow \quad \overline{z_1z_2} = \overline{8+i} = 8 - i \quad \text{...(i)}$$

Now, consider $\bar{z}_1\,\bar{z}_2 = (3-2i)(2+i)$

$$= 6 + 3i - 4i - 2i^2$$

$$= 6 - i - 2(-1) = 8 - i \quad [\because i^2 = -1] \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\overline{z_1z_2} = \bar{z}_1\,\bar{z}_2$$

EXAMPLE |9| If $z_1 = 3 + 5i$ and $z_2 = 2 - 3i$, then verify that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

Sol. Given, $z_1 = 3 + 5i$ and $z_2 = 2 - 3i$

Now, $$\frac{z_1}{z_2} = \frac{3+5i}{2-3i} = \frac{3+5i}{2-3i} \times \frac{2+3i}{2+3i}$$

[by rationalising the denominator]

$$= \frac{6 + 9i + 10i + 15i^2}{4 - 9i^2} = \frac{6 + 19i - 15}{4 + 9} \quad [\because i^2 = -1]$$

$$= \frac{-9 + 19i}{13} = \frac{-9}{13} + \frac{19}{13}i \quad \text{...(i)}$$

$$\therefore \text{LHS} = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{-9}{13} + \frac{19}{13}i\right)} = \frac{-9}{13} - \frac{19}{13}i$$

Now, consider RHS $= \frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{3+5i}}{\overline{2-3i}} = \frac{3-5i}{2+3i}$

$$= \frac{3-5i}{2+3i} \times \frac{2-3i}{2-3i}$$

[by rationalising the denominator]

$$= \frac{6 - 9i - 10i + 15i^2}{4 - 9i^2} = \frac{6 - 19i - 15}{4 + 9} \quad [\because i^2 = -1]$$

$$= \frac{-9 - 19i}{13} = \frac{-9}{13} - \frac{19}{13}i \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

EXAMPLE |10| If $x + iy = \frac{(a^2+1)^2}{2a-i}$, what is the value of $x^2 + y^2$?

Sol. We have, $x + iy = \frac{(a^2+1)^2}{2a-i}$ ...(i)

$\therefore \quad \overline{x+iy} = \overline{\left\{\frac{(a^2+1)^2}{2a-i}\right\}}$

$\Rightarrow \quad x - iy = \frac{(a^2+1)^2}{\overline{2a-i}}$ $\left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}\right]$

$\Rightarrow \quad x - iy = \frac{(a^2+1)^2}{2a+i}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(x+iy)(x-iy) = \frac{(a^2+1)^2(a^2+1)^2}{(2a-i)(2a+i)}$$

$\Rightarrow \quad x^2 - (iy)^2 = \frac{(a^2+1)^4}{(2a)^2 - i^2}$

$[\because (z_1+z_2)(z_1-z_2) = z_1^2 - z_2^2]$

$\Rightarrow \quad x^2 - i^2y^2 = \frac{(a^2+1)^4}{4a^2+1}$ $[\because i^2 = -1]$

$\therefore \quad x^2 + y^2 = \frac{(a^2+1)^4}{4a^2+1}$

EXAMPLE |11| If $x + iy = \sqrt{\frac{1+i}{1-i}}$, then prove that $x^2 + y^2 = 1$.

Sol. We have, $x + iy = \sqrt{\frac{1+i}{1-i}} \Rightarrow x + iy = \sqrt{\frac{1+i}{1-i} \times \frac{1+i}{1+i}}$

[by rationalising the denominator]

$\Rightarrow \quad x + iy = \sqrt{\frac{(1+i)^2}{1-i^2}}$ $[\because (z_1-z_2)(z_1+z_2) = z_1^2 - z_2^2]$

$\Rightarrow \quad x + iy = \frac{1+i}{\sqrt{1+1}} = \frac{1+i}{\sqrt{2}}$ $[\because i^2 = -1]$

$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ ...(i)

Now, taking conjugate on both sides, we get

$\overline{x+iy} = \overline{\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} \Rightarrow x - iy = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(x+iy)(x-iy) = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$\Rightarrow \quad x^2 - (iy)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{i}{\sqrt{2}}\right)^2$

$[\because (z_1+z_2)(z_1-z_2) = z_1^2 - z_2^2]$

$\Rightarrow x^2 - i^2y^2 = \frac{1}{2} - \frac{i^2}{2} \Rightarrow x^2 + y^2 = \frac{1}{2} + \frac{1}{2} = 1$ $[\because i^2 = -1]$

Hence proved.

MODULUS (ABSOLUTE VALUE) OF COMPLEX NUMBERS

The **modulus** (or absolute value) of a complex number, $z = a + ib$ is defined as the non-negative real number $\sqrt{a^2+b^2}$. It is denoted by $|z|$. i.e. $|z| = \sqrt{a^2+b^2}$

e.g. If $z = 2 + 3i$, then $|z| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

and if $z = 1 - i$, then $|z| = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$.

Knowledge Plus

(i) Multiplicative inverse of z is $\frac{\overline{z}}{|z|^2}$. It is also called reciprocal of z.

(ii) $z\overline{z} = |z|^2$

EXAMPLE |12| Find the modulus of the complex number $4 + 3i^7$.

Write the complex number in the form $z = a + ib$, then modulus of z is $|z| = \sqrt{a^2+b^2}$.

Sol. We have, $4 + 3i^7 = 4 + 3(i^4)(i^2)i$

$= 4 + 3(1)(-1)i$ $[\because i^4 = 1, i^2 = -1]$

$= 4 - 3i$

$\therefore$ Modulus $= |4 + 3i^7| = |4 - 3i|$

$= \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

EXAMPLE |13| Find the modulus of the complex number $\frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}}$.

Convert the complex number in the standard form and then find its modulus.

Sol. Let $z = \frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}} = \frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}} \times \frac{2\sqrt{3} + i\sqrt{2}}{2\sqrt{3} + i\sqrt{2}}$

[by rationalising the denominator]

$= \frac{6 + i\sqrt{6} - 2\sqrt{6}\,i - 2i^2}{(2\sqrt{3})^2 - (\sqrt{2}\,i)^2}$ $[\because (z_1-z_2)(z_1+z_2) = z_1^2 - z_2^2]$

$= \frac{6 - \sqrt{6}\,i + 2}{12 + 2}$ $[\because i^2 = -1]$

$= \frac{8 - \sqrt{6}\,i}{14} = \frac{8}{14} - \frac{\sqrt{6}}{14}i \Rightarrow z = \frac{4}{7} - \frac{\sqrt{6}}{14}i$

Now, modulus of z, $|z| = \sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{-\sqrt{6}}{14}\right)^2}$

$= \sqrt{\frac{16}{49} + \frac{6}{196}} = \sqrt{\frac{64+6}{196}} = \sqrt{\frac{70}{196}} = \sqrt{\frac{5}{14}}$

EXAMPLE |14| If $|z| = 1$, then find the value of $\frac{1+z}{1+\bar{z}}$.

Use the result $z\bar{z} = |z|^2$, then find its value.

Sol. Given, $|z| = 1 \Rightarrow |z|^2 = 1$

$\Rightarrow \quad z\bar{z} = 1 \quad [\because |z|^2 = z\bar{z}]$

Now, $\frac{1+z}{1+\bar{z}} = \frac{z\bar{z}+z}{1+\bar{z}} = \frac{z(\bar{z}+1)}{(\bar{z}+1)} = z \quad [\because 1 = z\bar{z}]$

EXAMPLE |15| Find the conjugate and modulus of the complex number $(1-i)^{-2} + (1+i)^{-2}$.

Sol. Let $z = (1-i)^{-2} + (1+i)^{-2}$

$$= \frac{1}{(1-i)^2} + \frac{1}{(1+i)^2} = \frac{(1+i)^2 + (1-i)^2}{(1-i)^2\,(1+i)^2}$$

$$= \frac{1+i^2+2i+1+i^2-2i}{(1-i^2)^2} = \frac{1-1+1-1}{(1+1)^2} = \frac{0}{4} \quad [\because i^2 = -1]$$

$$= 0 = 0 + 0i$$

$\therefore \quad \bar{z} = \overline{0+0i} = 0$ and $|z| = \sqrt{0+0} = 0$

EXAMPLE |16| Find the conjugate and modulus of the complex number $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$.

Sol. Let $z = \frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$

$$= \frac{(3+2i)(2+5i) + (3-2i)(2-5i)}{(2-5i)(2+5i)}$$

$$= \frac{6+15i+4i+10i^2+6-15i-4i+10i^2}{(2)^2-(5i)^2} \quad [\because (z_1-z_2)(z_1+z_2) = z_1^2 - z_2^2]$$

$$= \frac{6+20i^2+6}{4-25i^2} = \frac{12-20}{4+25} = \frac{-8}{29} = \frac{-8}{29} + 0i \quad [\because i^2 = -1]$$

Now, $\bar{z} = -\frac{8}{29} - 0i = \frac{-8}{29}$

and $|z| = \sqrt{\left(\frac{-8}{29}\right)^2 + 0^2} = \sqrt{\frac{64}{841}} = \frac{8}{29}$

EXAMPLE |17| If $|z| = 1$, then prove that $\frac{z-1}{z+1}$; $(z \neq 1)$ is a purely imaginary number. What will you conclude, if $z = 1$?

Sol. Let $z = a + ib$, such that $|z| = \sqrt{a^2+b^2} = 1$

$\Rightarrow \quad a^2 + b^2 = 1$

Now, consider $\left(\frac{z-1}{z+1}\right) = \left(\frac{a+ib-1}{a+ib+1}\right)$

$$= \frac{(a-1+ib)}{(a+1+ib)} \times \frac{(a+1-ib)}{(a+1-ib)}$$

[by rationalising the denominator]

$$= \frac{[(a-1)+ib]\,[(a+1)-ib]}{(a+1)^2-(ib)^2} \quad [\because (z_1+z_2)(z_1-z_2) = z_1^2 - z_2^2]$$

$$= \frac{a^2-1-iab+ib+iab+ib-i^2b^2}{(a+1)^2-i^2b^2}$$

$$= \frac{(a^2+b^2-1)+2bi}{(a+1)^2+b^2} \quad [\because i^2 = -1]$$

$$= \frac{(1-1)+2bi}{a^2+1+2a+b^2}$$

$[\because a^2+b^2 = 1$ and $(z_1+z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2]$

$$= \frac{0+2bi}{(a^2+b^2)+1+2a} = \frac{2bi}{2+2a} \quad [\because a^2+b^2 = 1]$$

$$= 0 + \frac{bi}{1+a}$$

Clearly, real part of $\left(\frac{z-1}{z+1}\right)$ is zero and imaginary part of $\left(\frac{z-1}{z+1}\right)$ is $\frac{b}{1+a}$.

$\therefore \quad \left(\frac{z-1}{z+1}\right) = \frac{ib}{1+a}$ is purely imaginary.

Again, when $z = 1$, then

$\left(\frac{z-1}{z+1}\right) = \frac{1-1}{1+1} = 0$, which is purely real.

EXAMPLE |18| Find the complex number satisfying the equation $z + \sqrt{2}|(z+1)| + i = 0$. **[NCERT Exemplar]**

Sol. We have, $z + \sqrt{2}|(z+1)| + i = 0$

Let $z = x + iy$.

Then, $(x+iy) + \sqrt{2}|(x+iy+1)| + i = 0$

$\Rightarrow \quad x + i(y+1) + \sqrt{2}|(x+1)+iy| = 0$

$\Rightarrow \quad x + i(y+1) + \sqrt{2}\sqrt{(x+1)^2+y^2} = 0$

[if $z = a+ib$, then $|z| = \sqrt{a^2+b^2}$]

$\Rightarrow x + \sqrt{2}\sqrt{x^2+1+2x+y^2} + i(y+1) = 0 + 0i$

On equating real and imaginary part, we get

$$x + \sqrt{2}\sqrt{x^2+1+2x+y^2} = 0 \quad \text{...(i)}$$

and $$y + 1 = 0 \quad \text{...(ii)}$$

From Eq. (ii), we get

$$y = -1$$

Now, on substituting $y = -1$ in Eq. (i), we get

$$x + \sqrt{2}\sqrt{x^2+1+2x+1} = 0$$

$\Rightarrow \quad x = -\sqrt{2}\sqrt{x^2+2x+2}$

On squaring both sides, we get

$$x^2 = 2(x^2 + 2x + 2)$$

$$\Rightarrow \quad x^2 = 2x^2 + 4x + 4 \Rightarrow x^2 + 4x + 4 = 0$$

$$\Rightarrow \quad (x+2)^2 = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$$

Hence, $z = x + iy = -2 - i$

Properties of Modulus of Complex Numbers

1. $|z| = |\bar{z}|$
2. $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$
3. $-|z| \le \text{Re}(z) \le |z|$
4. $-|z| \le \text{Im}(z) \le |z|$
5. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$
6. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$
7. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
8. $|z_1 z_2| = |z_1||z_2|$

   In general, if $z_1, z_2, \ldots, z_n$ are any complex numbers, then

   $$|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n| \quad \ldots(i)$$

   So, if $z_1 = z_2 = \ldots = z_n$, then from Eq. (i), we have $|z_1^n| = |z_1|^n$ Thus, we have $|z^n| = |z|^n$.
9. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \ne 0$
10. $|z_1 + z_2| \le |z_1| + |z_2|$
11. $|z_1 - z_2| \ge |z_1| - |z_2|$

Note Property 10 and 11 are called triangle inequality.

> ***Knowledge Plus***
>
> In the set of complex number, $z_1 > z_2$ or $z_1 < z_2$ are meaningless but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

EXAMPLE |19| If $z_1 = 3 + 2i$ and $z_2 = 1 - 3i$, then find the modulus of z_1 and z_2.

Also, verify that $|z_1 z_2| = |z_1||z_2|$.

Sol. Given, $z_1 = 3 + 2i$ and $z_2 = 1 - 3i$

Clearly, $|z_1| = \sqrt{(3)^2 + (2)^2} = \sqrt{9+4} = \sqrt{13}$

and $|z_2| = \sqrt{1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$

$\therefore \quad |z_1||z_2| = \sqrt{13}\sqrt{10} = \sqrt{130} \quad \ldots(i)$

Now, consider $z_1 z_2 = (3+2i)(1-3i) = 3 - 9i + 2i - 6(i^2)$

$= 3 - 7i + 6 = 9 - 7i \quad [\because i^2 = -1]$

$\therefore \quad |z_1 z_2| = \sqrt{9^2 + (-7)^2}$

$= \sqrt{81+49} = \sqrt{130} \quad \ldots(ii)$

From Eqs. (i) and (ii), we get $|z_1 z_2| = |z_1||z_2|$

EXAMPLE |20| If $z_1 = 3 + 2i$ and $z_2 = 2 - 4i$, then verify that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

Sol. We have, $z_1 = 3 + 2i$ and $z_2 = 2 - 4i$

Now, $\text{LHS} = |z_1 + z_2|^2 + |z_1 - z_2|^2$

On substituting the values of z_1 and z_2, we get

$\text{LHS} = |3 + 2i + 2 - 4i|^2 + |3 + 2i - 2 + 4i|$

$= |5 - 2i|^2 + |1 + 6i|^2 = (5)^2 + (-2)^2 + (1)^2 + (6)^2$

$[\text{if } z = a + ib, \text{ then } |z|^2 = a^2 + b^2]$

$= 25 + 4 + 1 + 36$

$\therefore \quad \text{LHS} = 66 \quad \ldots(i)$

and $\text{RHS} = 2(|z_1|^2 + |z_2|^2) = 2(|3+2i|^2 + |2-4i|^2)$

$= 2[(3)^2 + (2)^2 + (2)^2 + (-4)^2]$

$= 2(9 + 4 + 4 + 16) = 2 \times 33$

$\therefore \quad \text{RHS} = 66 \quad \ldots(ii)$

From Eq. (i) and (ii), we get

LHS = RHS **Hence proved.**

EXAMPLE |21| If $z_1 = 3 + i$ and $z_2 = 1 + 4i$, then verify that $|z_1 + z_2| < |z_1| + |z_2|$.

Sol. We have, $z_1 = 3 + i$ and $z_2 = 1 + 4i$

$\therefore \quad |z_1| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$

and $|z_2| = \sqrt{1^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$

$\therefore \quad |z_1| + |z_2| = \sqrt{10} + \sqrt{17}$

$= 3.16 + 4.12 = 7.28 \quad \ldots(i)$

Now, $z_1 + z_2 = 3 + i + 1 + 4i = 4 + 5i$

$\therefore \quad |z_1 + z_2| = \sqrt{4^2 + 5^2} = \sqrt{16+25} = \sqrt{41} = 6.40 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get $|z_1 + z_2| < |z_1| + |z_2|$

EXAMPLE |22| If $z_1 = 2 - i$ and $z_2 = 1 + i$, then find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$. **[NCERT]**

Sol. We have,

$z_1 = 2 - i$ and $z_2 = 1 + i$

$$\therefore \quad \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{2 - i + 1 + i + 1}{2 - i - 1 - i + 1}\right|$$

$$= \left|\frac{4}{2 - 2i}\right| = \left|\frac{2}{1-i}\right| = \frac{2}{|1-i|} \quad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$= \frac{2}{\sqrt{(1)^2 + (-1)^2}} = \frac{2}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

EXAMPLE |23| Find $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right|$. **[NCERT Exemplar]**

Sol. Let $z = \dfrac{(1+i)(2+i)}{(3+i)} = \dfrac{2+i+2i+i^2}{3+i} = \dfrac{2+3i-1}{3+i}$

$\Rightarrow \quad z = \dfrac{1+3i}{3+i} \qquad [\because i^2 = -1]$

Now, $|z| = \left|\dfrac{1+3i}{3+i}\right| = \dfrac{|1+3i|}{|3+i|} \qquad \left[\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right]$

$= \dfrac{\sqrt{1^2+3^2}}{\sqrt{3^2+1^2}} = 1$

Hence, $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right| = 1$

EXAMPLE |24| If z_1, z_2 are complex numbers such that $\dfrac{4z_1}{5z_2}$ is purely imaginary number, then find $\left|\dfrac{z_1-z_2}{z_1+z_2}\right|$.

Sol. Since, $\dfrac{4z_1}{5z_2}$ is purely imaginary number.

$\therefore \quad \dfrac{4z_1}{5z_2} = \lambda i$ for some $\lambda \in R \Rightarrow \dfrac{z_1}{z_2} = \dfrac{5\lambda}{4} i \qquad$ ...(i)

Now, consider $\left|\dfrac{z_1-z_2}{z_1+z_2}\right| = \left|\dfrac{\dfrac{z_1}{z_2}-1}{\dfrac{z_1}{z_2}+1}\right|$

[dividing numerator and denominator by z_2]

$= \left|\dfrac{\dfrac{5}{4}\lambda i - 1}{\dfrac{5}{4}\lambda i + 1}\right| = \left|\dfrac{5\lambda i - 4}{5\lambda i + 4}\right| \qquad$ [using Eq. (i)]

$= \dfrac{|5\lambda i - 4|}{|5\lambda i + 4|} = \dfrac{|-4+5\lambda i|}{|4+5\lambda i|} = \dfrac{\sqrt{(-4)^2+(5\lambda)^2}}{\sqrt{(-4)^2+(5\lambda)^2}} = 1$

Hence, $\left|\dfrac{z_1-z_2}{z_1+z_2}\right| = 1$

EXAMPLE |25| If $(2+i)\ (2+2i)\ (2+3i)\ \ldots(2+ni) = x+iy$, then prove that $5\cdot 8\cdot 13 \ldots (4+n^2) = x^2+y^2$.

[NCERT Exemplar]

Sol. We have, $(2+i)(2+2i)(2+3i)\ldots(2+ni) = x+iy$

On taking modulus both sides, we get

$|(2+i)(2+2i)(2+3i)\ldots(2+ni)| = |x+iy|$

$\Rightarrow \quad |2+i||2+2i|\ldots|2+ni| = |x+iy|$

$[\because |z_1 z_2 \ldots z_n| = |z_1|\cdot|z_2|\ldots|z_n|]$

$\Rightarrow (\sqrt{4+1})(\sqrt{4+4})\ldots(\sqrt{4+n^2}) = \sqrt{x^2+y^2}$

On squaring both sides, we get

$5\cdot 8 \ldots (4+n^2) = x^2+y^2$ **Hence proved.**

ARGAND PLANE

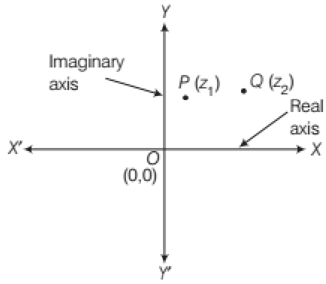
A complex number $z = a+ib$ can be represented by a unique point $P(a, b)$ in the cartesian plane referred to a pair of rectangular axes. A purely real number a, i.e. $(a+0i)$ is represented by the point $(a, 0)$ on X-axis. Therefore, X-axis is called real axis.

A purely imaginary number ib i.e. $(0+ib)$ is represented by the point $(0, b)$ on Y-axis. Therefore, Y-axis is called **imaginary axis.** The intersection (common) of two axes is called zero complex number i.e. $z = 0+0i$.

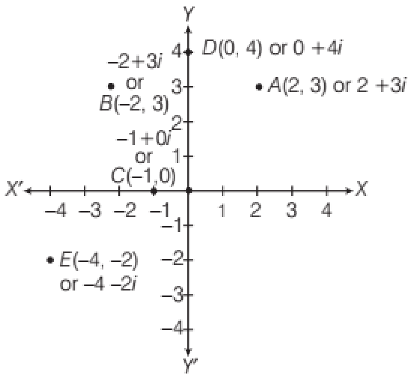
Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram**. The plane representing complex numbers as points, is called **Complex plane or Argand plane or Gaussian plane.**

If two complex numbers z_1 and z_2 are represented by the points P and Q in the complex plane, then

$$|z_1 - z_2| = PQ = \text{Distance between } P \text{ and } Q$$



e.g. The complex numbers such as $2+3i, -2+3i, -1+0i, 0+4i$ and $-4-2i$ which correspond to the ordered pairs $(2, 3), (-2, 3), (-1, 0), (0, 4)$ and $(-4, -2)$ respectively, can be represented geometrically by the points A, B, C, D and E respectively, in the cartesian plane, as shown in the figure.



EXAMPLE |26| If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3}+i$, then find the quadrant in which $\left(\dfrac{z_1}{z_2}\right)$ lies. **[NCERT Exemplar]**

Sol. We have, $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$.

$$\therefore \quad \frac{z_1}{z_2} = \frac{\sqrt{3}(1+i)}{\sqrt{3}+i} = \frac{\sqrt{3}(1+i)}{(\sqrt{3}+i)} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$$

[by rationalising the denominator]

$$= \frac{\sqrt{3}(1+i)(\sqrt{3}-i)}{(\sqrt{3})^2 - (i)^2} \quad [\because (z_1+z_2)(z_1-z_2) = z_1^2 - z_2^2]$$

$$= \frac{\sqrt{3}(\sqrt{3} - i + i\sqrt{3} - i^2)}{3 - i^2}$$

$$= \frac{\sqrt{3}(\sqrt{3} + i(\sqrt{3}-1)+1)}{3+1} \quad [\because i^2 = -1]$$

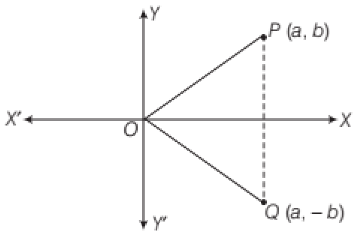
$$= \frac{\sqrt{3}}{4}((\sqrt{3}+1) + i(\sqrt{3}-1))$$

$$= \frac{\sqrt{3}(\sqrt{3}+1)}{4} + \frac{i\sqrt{3}(\sqrt{3}-1)}{4}$$

which is represented by a point in first quadrant.

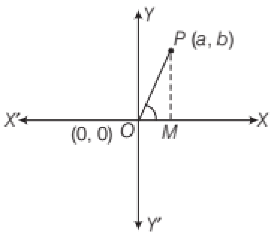
Representation of Conjugate of z on Argand Plane

Geometrically, the mirror image of the complex number $z = a + ib$ (represented by the ordered pair (a, b) about the X-axis is called **conjugate of z** which is represented by the ordered pair $(a, -b)$. If $z = a + ib$, then $\bar{z} = a - ib$.



Representation of Modulus of z on Argand Plane

Geometrically, the distance of the complex number $z = a + ib$ [represented by the ordered pair (a, b)] from origin, is called the modulus of z.



$$\therefore \quad OP = \sqrt{(a-0)^2 + (b-0)^2}$$

$$= \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2} = |a + ib|$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1. The conjugate of a complex number $z = a + ib$, is

(a) $\bar{z} = a + ib$ (b) $\bar{z} = a - ib$
(c) $\bar{z} = ia + b$ (d) $\bar{z} = ia - b$

2. Which of the following are correct?

I. $|3 + i| = \sqrt{10}$; $|2 - 5i| = \sqrt{29}$

II. $\overline{(3+i)} = 3 - i$; $\overline{(2-5i)} = 2 + 5i$ and $\overline{(-3i-5)} = 3i - 5$

III. $z^{-1} = \frac{\bar{z}}{|z|^2}$ or $Z\bar{Z} = |z|^2, z \neq 0$

(a) I and III are correct (b) I and II are correct
(c) All are correct (d) None of these

3. If $|1 - i|^n = 2^n$, then n is equal to

(a) 1 (b) 0
(c) –1 (d) None of these

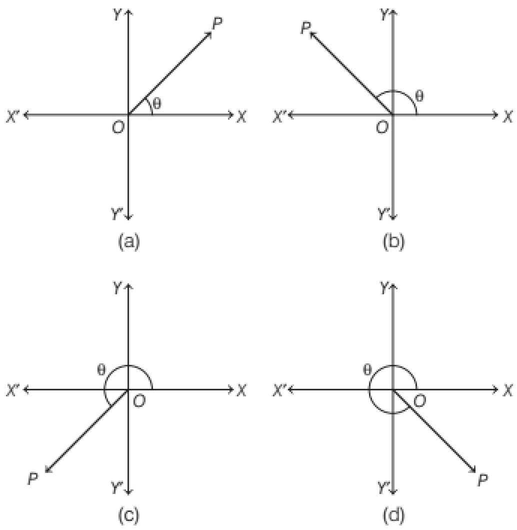
4 The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

(a) $|z+3|^2$ (b) $|z-3|$
(c) $z^2 + 3$ (d) None of these

5 If $a + ib = c + id$, then

(a) $a^2 + c^2 = 0$ (b) $b^2 + c^2 = 0$
(c) $b^2 + d^2 = 0$ (d) $a^2 + b^2 = c^2 + d^2$

6 The geometrical representation of complex number $z = \frac{-16}{1 + i\sqrt{3}}$, is



VERY SHORT Type Questions

7 Find the conjugate of the complex numbers.
(i) $-i\sqrt{5}$ (ii) $\sqrt{3}$

8 Find the complex conjugates of
(i) $2+i5$ (ii) $-6-i7$ (iii) $\sqrt{3}$

9 Find the multiplicative inverse of the complex number $\sqrt{5}+3i$. [NCERT]

10 If $(1+i)z=(1-i)\bar{z}$, then show that $z=-i\bar{z}$. [NCERT Exemplar]

11 Find the modulus of the conjugate of the complex number $-3i$.

12 Find the number of non-zero integral solutions of the equation $|1-i|^x=2^x$.

SHORT ANSWER Type Questions

13 If $z_1=\sqrt{2}-3i$ and $z_2=5-i\sqrt{2}$, then find the quadrant in which $\frac{z_1}{z_2}$ lies.

14 Find the conjugate of the complex number $\frac{1-i}{1+i}$. [NCERT Exemplar]

15 Find the conjugate of $(6+5i)^2$.

16 Find the real numbers x and y, if $(x-iy)(3+5i)$ is the conjugate of $(-1-3i)$.

17 Find the conjugate and modulus of the complex number $(3-2i)(3+2i)(1+i)$.

18 If $z=12+5i$, then verify that
(i) $(\bar{z})=z$ (ii) $z+\bar{z}=2\text{Re}(z)$

19 Find the modulus of the complex number $4+3i^7$.

20 Find the conjugate and modulus of the complex number $\frac{2+3i}{3+2i}$.

21 If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$, then show that

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)=A^2+B^2.$$

[NCERT]

LONG ANSWER Type I Questions

22 Find the conjugate of

$$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}.$$

[NCERT]

23 Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i are conjugate of each other.

24 Find all non-zero complex numbers z satisfying $\bar{z}=iz^2$.

25 If $x+iy=\frac{a+ib}{a-ib}$, prove that $x^2+y^2=1$. [NCERT]

26 If $a+ib=\frac{(x^2+1)}{2x^2+1}$, prove that

$$a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}.$$

[NCERT]

27 Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$.

28 If $z=12-5i$, then verify that
(i) $-|z|\le\text{Re}(z)\le|z|$
(ii) $-|z|\le\text{Im}(z)\le|z|$

29 If $z_1=3+i$ and $z_2=1+4i$, then verify that $|z_1-z_2|\ge|z_2|-|z_1|$.

30 If $f(z)=\frac{7-z}{1-z^2}$, where $z=1+2i$, then find $|f(z)|$. [NCERT Exemplar]

31 If $\frac{z-1}{z+1}$ is a purely imaginary number $(z\ne-1)$, then find the value of $|z|$. [NCERT Exemplar]

32 If $|z+1|=z+2(1+i)$, then find z.

33 If $z=x+iy$, $w=\frac{1-iz}{z-i}$ and $|w|=1$, then show that z is purely real.

34 If z is a complex number such that $|z-1|=|z+1|$, then show that $\text{Re}(z)=0$.

HINTS & ANSWERS

1. (b) By definition, $\bar{z} = a - ib$.

2. (c) I. $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}, |2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$

II. $\overline{(3+i)} = 3 - i, \overline{(2-5i)}, = 2+5i, \overline{(-3i-5)} = 3i - 5$

III. The multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$$

$$= \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

$$\therefore\ z^{-1} = \frac{\bar{z}}{|z|^2} \text{ or } z\bar{z} = |z|^2$$

3. (b) We know that, if two complex numbers are equal then their modulus must also be equal.

$$|1-i|^n = 2^n$$

$$\Rightarrow (\sqrt{2})^n = 2^n \qquad [\because |1-i| = \sqrt{2}]$$

$$\Rightarrow 2^{n/2} = 2^n$$

$$\Rightarrow \frac{n}{2} = n$$

$$\Rightarrow n = 0$$

4. (a) Let $z = x + iy$

Then, $(z+3)(\bar{z}+3) = (x+iy+3)(x+3-iy)$

$$= (x+3)^2 - (iy)^2 = (x+3)^2 + y^2$$

$$= |x+3+iy|^2 = |z+3|^2$$

5. (d) If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then

$$|z_1| = |z_2|$$

$$\Rightarrow \sqrt{x_1^2+y_1^2} = \sqrt{x_2^2+y_2^2}$$

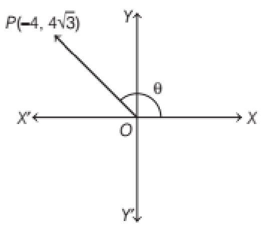
6. (b) We have, $z = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$

$$= \frac{-16(1-i\sqrt{3})}{1^2-(i\sqrt{3})^2}$$

$$= \frac{-16(1-i\sqrt{3})}{1+3}$$

$$= -4+4i\sqrt{3},$$

which can be represented geometrically as shown below.



7. (i) $z = 0 - i\sqrt{5} \Rightarrow \bar{z} = 0 + i\sqrt{5}$ **Ans.** $i\sqrt{5}$

(ii) $\overline{\sqrt{3}}$ **Ans.** $-6+7i$

8. (i) $\overline{2+i5} = 2 - i5$

Ans. $2-5i$ (ii) $\overline{-6-i7} = -6+i7$

(iii) $\overline{\sqrt{3}} = \overline{\sqrt{3}+0i} = \sqrt{3} - 0i$ **Ans.** $\sqrt{3}$

9. Use formula, multiplication of $z = \frac{\bar{z}}{|z|^2}$ **Ans.** $\frac{\sqrt{5}}{14} - \frac{3}{14}i$

10. We have, $(1+i)z = (1-i)\bar{z}$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1-1-2i}{1+1}$$

11. $z = 3i \Rightarrow z = 3i$ **Ans.** 3

12. We have, $(\sqrt{1^2+(-1)^2})^x = 2^x \Rightarrow 2^{x/2} = 2^x \Rightarrow \frac{x}{2} = x$

Ans. $x = 0$

13. Solve as Example 26. **Ans.** IVth quadrant

14. $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1-1-2i}{1+1} = -i$ **Ans.** i

15. $z = (6+5i)^2 = 36 - 25 + 60i = 11 + 60i$ **Ans.** $11 - 60i$

16. Solve as Example 4. **Ans.** $x = \frac{6}{17}$ and $y = -\frac{7}{17}$

17. Let $z = (3-2i)(3+2i)(1+i)$

$$\therefore\ z = (9+6i-6i-4i^2)(1+i)$$

$$= (9+4)(1+i) = 13+13i$$

Ans. $\bar{z} = 13 - 13i$ and $|z| = 13\sqrt{2}$

19. $z = 4+3i^4i^3 = 4-3i$ **Ans.** 5

20. $z = \frac{2+3i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{12+5i}{13}$

Ans. $\bar{z} = \frac{12}{13} - \frac{5}{13}i$ and $|z| = 1$

21. Solve as Example 25.

22. $z = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{63-16i}{16+9}$ **Ans.** $\frac{63}{25} + \frac{16}{25}i$

23. Since, $-3+ix^2y$ and x^2+y+4i are conjugate of each other

$$\therefore\ -3+ix^2y = \overline{x^2+y+4i}$$

After this, equate real and imaginary parts, to get the values of x and y.

Ans. $(x = 1, y = -4)$

or $(x = -1, y = -4)$

24. Solve as Example 7.

Ans. $0+0i, 0+i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

25. We have, $x + iy = \dfrac{a + ib}{a - ib}$ ...(i)

Take modulus both sides of Eq. (i) and then solve it.

26. We have, $a + ib = \dfrac{x^2 + 1}{2x^2 + 1}$ ...(i)

Take modulus both sides of Eq. (i) and then solve it.

27. $\dfrac{(1+i)^2 - (1-i)^2}{1^2 - i^2} = \dfrac{4i}{2} = 2i$ **Ans.** 2

28. **Hint** $|z| = 13$

$\text{Re}(z) = 12$

$\text{Im}(z) = -5$

29. $|z_1 - z_2| = |2 - 3i| = \sqrt{4+9} = \sqrt{13}$

$|z_1| = \sqrt{9+1} = \sqrt{10}$ and $|z_2| = \sqrt{1+16} = \sqrt{17}$

30. $\dfrac{2-i}{2}$

31. Let $z = x + iy$, then

$$\frac{z-1}{z+1} = \frac{(x^2-1) + y^2 + i[y(x+1) - y(x-1)]}{(x^2+1)^2 + y^2}$$

$\because$ $\dfrac{z-1}{z+1}$ is purely imaginary.

$\therefore$ $\text{Re}\left(\dfrac{z-1}{z+1}\right) = 0$, i.e. $\dfrac{(x^2-1) + y^2}{(x+1)^2 + y^2} = 0$

$$x^2 - 1 + y^2 = 0 \Rightarrow x^2 + y^2 = 1$$

Ans. $|z| = 1$

32. Let $z = x + iy$, then

$$|x + iy + 1| = x + iy + 2(1+i)$$

$$\Rightarrow \sqrt{(x+1)^2 + y^2} = x + 2 + i(y+2)$$

Ans. $z = \dfrac{1}{2} - 2i$

33. We have,

$$|w| = 1$$

$$\Rightarrow \frac{|1 - iz|}{|z - i|} = 1$$

$$\Rightarrow |1 - iz| = |z - i|$$

$$\Rightarrow |1 + y - ix| = |x + i(y-1)|$$

34. Let $z = x + iy$, then $|z - 1| = |z + 1|$

$$\Rightarrow |x + iy - 1| = |x + iy + 1|$$

$$\Rightarrow |(x-1) + iy| = |(x+1) + iy|$$

$$\Rightarrow \sqrt{(x-1)^2 + y^2} = \sqrt{(x+1)^2 + y^2}$$

$$\Rightarrow (x-1)^2 + y^2 = (x+1)^2 + y^2$$

$$\Rightarrow x^2 + 1 - 2x = x^2 + 1 + 2x$$

$$\Rightarrow 4x = 0$$

$$\Rightarrow x = 0$$

$$\therefore \text{Re}(z) = 0$$

SUMMARY

- A number consisting of real number and imaginary number is called **complex number**, i.e. $z = a + ib$, where a is **real part** Re(z) and b is **imaginary part** Im(z).
- A complex number $z = a + ib$ is called purely real, if $b = 0$, i.e. Im(z) = 0 and is called purely imaginary, if $a = 0$, i.e. Re(z) = 0.
- **Integral Powers of *i***

  (i) $i^{4q} = 1, q \in N$ (ii) $i^{4q+1} = i, q \in N$ (iii) $i^{4q+2} = -1, q \in N$ (iv) $i^{4q+3} = -i, q \in N$

  (v) $i^{-p} = \frac{1}{i^p}, p \in N$
- Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal, if $a = c$ and $b = d$.
- **Algebra of Complex Numbers** Let two complex numbers are $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then their

  (i) ***Addition*** (sum) is defined as

  $$z = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

  (ii) ***Subtraction*** $z_1 - z_2$ is defined as the addition of z_1 and $(-z_2)$

  i.e. $z_1 - z_2 = z_1 + (-z_2) = (a_1 - a_2) + i(b_1 - b_2)$.

  (iii) ***Multiplication*** is defined as $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bd)$

  (iv) ***Division*** $\frac{z_1}{z_2}$ is defined as the multiplication of z_1 by the multiplicative inverse of z_2

  i.e. $$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = \left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}\right)$$
- The conjugate $\bar{z}$ of a complex number z, is the complex number obtained by changing the sign of imaginary part of z.
- The modulus $|z|$ (or absolute value) of a complex number $z = a + ib$ is defined as the non-negative real number.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. If $x = \sqrt{-16}$, then

(a) $x = 4i$ (b) $x = 4$
(c) $x = -4$ (d) All of these

2. Which of the following is true?

(a) $1 - i < 1 + i$ (b) $2i + 1 > -2i + 1$
(c) $2i > 1$ (d) None of these

3. If $z + 0 = z$, where $z = x + i\,y$ and $0 = 0 + i\,0$, then 0 is called

(a) additive identity (b) additive inverse
(c) closure (d) None of these

4. If $z_1 = 2 + 3\,i$ and $z_2 = = 3 - 2\,i$, then $z_1 - z_2$ is equal to

(a) $-1 + 5\,i$ (b) $5 - i$
(c) $i + 5$ (d) None of these

5. If $z = 5i\left(-\frac{3}{5}i\right)$, then z is equal to

(a) $0 + 3\,i$ (b) $3 + 0i$ (c) $0 - 3i$ (d) $-3 + 0i$

6. If $z = i^9 + i^{19}$, then z is equal to

(a) $0 + 0i$ (b) $1 + 0i$ (c) $0 + i$ (d) $1 + 2i$

7. If $z \neq 0$ is a complex number, then

(a) $\text{Re}(z) = 0 \Rightarrow \text{Im}(z^2) = 0$
(b) $\text{Re}(z^2) = 0 \Rightarrow \text{Im}(z^2) = 0$
(c) $\text{Re}(z) = 0 \Rightarrow \text{Re}(z^2) = 0$
(d) None of the above

VERY SHORT ANSWER Type Questions

8. Find the values of x and y, if $x + 4iy = ix + y + 3$.

9. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

10. Find the value of $1 + i^2 + i^4 + i^6 + \dots + i^{20}$.

[NCERT Exemplar]

11. Find the value of i^{-1097}.

12. Prove that $\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right)$ is purely real.

13. Express $\left(-2 - \frac{1}{3}i\right)^3$ in the form $a + ib$.

14. Express $\frac{1}{-2+\sqrt{-3}}$ in the form $a + ib$.

15. Express $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ in the form $a + ib$.

SHORT ANSWER Type I Questions

16. Express $\frac{1}{1 - \cos\theta + 2i\sin\theta}$ in the form $a + ib$.

17. Find the smallest positive integral value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

18. What is the reciprocal of $3 + \sqrt{7}i$?

19. Find the multiplicative inverse of $1 + i$.

20. Find the quadrant in which conjugate of $\frac{1+2i}{1-i}$ lies.

SHORT ANSWER Type II Questions

21. If $z = 2 - 3\,i$, show that $z^2 - 4z + 13 = 0$ and hence find the value of $4z^3 - 3z^2 + 169$.

22. If $(1+i)(1+2i)(1+3i)\dots(1+ni) = (x+iy)$, then show that $2\cdot5\cdot10\ \dots\ (1+n^2) = x^2 + y^2$.

23. If $|z_1| = |z_2| = \dots = |z_n| = 1$, then show that $|z_1 + z_2 + \dots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n}\right|$.

CASE BASED Questions

24. A complex number z is pure real if and only if $\bar{z} = z$ and is pure imaginary if and only if $\bar{z} = -z$.

Based on the above information answer the following questions.

(i) If $(1+i)z = (1-i)\bar{z}$, then $-i\bar{z}$ is

(a) $-\bar{z}$ (b) z (c) $\bar{z}$ (d) z^{-1}

(ii) $\overline{z_1 z_2}$ is

(a) $\bar{z}_1\,\bar{z}_2$ (b) $\bar{z}_1 + \bar{z}_2$ (c) $\frac{\bar{z}_1}{\bar{z}_2}$ (d) $\frac{1}{\bar{z}_1\,\bar{z}_2}$

(iii) If x and y are real numbers and the complex number $\frac{(2+i)x-i}{4+i}+\frac{(1-i)y+2i}{4i}$ is pure real, the relation between x and y is

(a) $8x-17y=16$ (b) $8x+17y=16$

(c) $17x-8y=16$ (d) $17x-8y=-16$

(iv) If $z=\frac{3+2i\sin\theta}{1-2i\sin\theta}\left(0<\theta\leq\frac{\pi}{2}\right)$ is pure imaginary, then θ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{12}$

(v) If z_1 and z_2 are complex numbers such that

$$\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$$

(a) $\frac{z_1}{z_2}$ is pure real

(b) $\frac{z_1}{z_2}$ is pure imaginary

(c) z_1 is pure real

(d) z_1 and z_2 are pure imaginary

HINTS & ANSWERS

1. (a) Here, $x=\sqrt{-16}$

$x=\sqrt{-1\times16}$

$=\sqrt{-1}\times\sqrt{4\times4}=4i$

2. (d) Since, comparison of complex numbers is not valid.

3. (a) For every complex number z, we have a complex number $0+i0$ (denoted by 0) called additive identity or zero complex number such that $z+0=z$.

4. (a) Here, $z_1=2+3i$, $z_2=3-2i$, then

$z_1-z_2=2+3i-(3-2i)$

$=2+3i-3+2i=-1+5i$

5. (b) $5i\left(-\frac{3}{5}i\right)=5\times-\frac{3}{5}i^2=-3(-1)=3=3+0i$

6. (a) $i^9+i^{19}=i^9(1+i^{10})=i^9[1+(i^2)^5]$ (taking i^9 common)

$=i^9[1+(-1)^5]=i^9(1-1)=0=0+0i$

7. (a) Let $z=x+iy$

If $\text{Re}(z)=0$, then $z=iy$

$z^2=(iy)^2=-y^2$

Thus, $z^2=-y^2$ (which is real)

$\Rightarrow$ $\text{Im}(z^2)=0$

8. $x+4iy=ix+y+3 \Rightarrow x=y+3$ and $4y=x$

Ans. $x=4$ and $y=1$

9. $1+i^{10}+i^{20}+i^{30}=1+(i^4)^2i^2+(i^4)^5+(i^4)^7i^2$

$=1-1+1-1=0$

10. $1+i^2+i^4+...+i^{20}=\frac{1((i^2)^{11}-1)}{(i^2)-1}=\frac{1(-1-1)}{-1-1}$ **Ans.** 1

11. $i^{-1097}=\frac{1}{i^{4\times274+1}}=\frac{1}{i}\times\frac{i}{i}$ **Ans.** $-i$

12. $\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right)=\frac{(2)^2-(3i)^2}{(3)^2-(4i)^2}$

$=\frac{4+9}{9+4}=1$

13. $\left(-2-\frac{1}{3}i\right)^3=-\left[(2)^3+\left(\frac{1}{3}i\right)^3+3\times2^2\times\left(\frac{i}{3}\right)+3\times2\times\left(\frac{i}{3}\right)^2\right]$

$=-\left[8-\frac{i}{27}+4i-\frac{2}{3}\right]$ **Ans.** $-\frac{22}{3}-\frac{107}{27}i$

14. $\frac{1}{-2+\sqrt{3}i}\times\frac{-2-\sqrt{3}i}{-2-\sqrt{3}i}=\frac{-2-\sqrt{3}i}{(-2)^2-(\sqrt{3}i)^2}=\frac{-2-\sqrt{3}i}{4+3}$

Ans. $-\frac{2}{7}-\frac{\sqrt{3}}{7}i$

15. $\frac{2-\sqrt{25}i}{1-\sqrt{16}i}=\frac{2-5i}{1-4i}\times\frac{1+4i}{1+4i}=\frac{22+3i}{17}$ **Ans.** $\frac{22}{17}+\frac{3}{17}i$

16. $z=\frac{1}{(1-\cos\theta)+2-\sin\theta}\times\frac{(1-\cos\theta)-2i\sin\theta}{(1-\cos\theta)-2i\sin\theta}$

$=\frac{(1-\cos\theta)-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}=\frac{(1-\cos\theta)-2i\sin\theta}{1+\cos^2\theta-2\cos\theta+4\sin^2\theta}$

Ans. $\left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right)+i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$

17. $z=\left(\frac{1+i}{1-i}\right)^n\times(1-i)^2=\left[\left(\frac{(1+i)^2}{1^2-i^2}\right)\right]^n(1^2+i^2-2i)$

$=\left(\frac{1-1+2i}{2}\right)^n(-2i)=(i)^n(-2i)$ **Ans.** 1

18. $z=3+\sqrt{7}i$

$\therefore$ $\frac{1}{z}=\frac{1}{3+\sqrt{7}i}\times\frac{3-\sqrt{7}i}{3-\sqrt{7}i}=\frac{3-\sqrt{7}i}{9+7}$ **Ans.** $\frac{3}{16}-\frac{\sqrt{7}}{16}i$

19. $z=1+i$

$\therefore$ Multiplicative inverse $=\frac{1}{z}=\frac{1}{1+i}\times\frac{1-i}{1-i}=\frac{1-i}{2}$

Ans. $\frac{1}{2}-\frac{i}{2}$

20. $z=\frac{1+2i}{1-i}\times\frac{1+i}{1+i}=\frac{-1+3i}{1+1}=\frac{1}{2}(-1+3i)$

$\therefore$ $\bar{z}=\frac{1}{2}(-1-3i)$

Ans. IIIrd quadrant

21. Now, $z^2 = (2-3i)^2 = 4-9-12i = -5-12i$ and $z^3 = (2-3i)^3$

$$= (2)^3 - (3i)^3 - 3(2)^2(3i) + 3(2)(3i)^2$$

$$= 8 + 27i - 36i - 54 = -46 - 9i$$

Now, $z^2 - 4z + 13 = (-5-12i) - 4(2-3i) + 13 = 0$

and $4z^3 - 3z^2 + 169 = -46 - 9i - 3(-5-12i) + 169$

Ans. $138 + 27i$

22. $|(1+i)(1+2i)(1+3i)...(1+ni)| = |x+iy|$

$\Rightarrow |1+i||1+2i||1+3i|...|1+ni| = |x+iy|$

$\Rightarrow \sqrt{1+1}\sqrt{1+4}\sqrt{1+9}...\sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow \sqrt{2}\sqrt{5}\sqrt{10}...\sqrt{1+n^2} = \sqrt{x^2+y^2}$

23. Given, $|z_1| = |z_2| = ... = |z_n| = 1$

$\Rightarrow |z_1|^2 = |z_2|^2 = ... = |z_n|^2 = 1$

$\therefore z_1\bar{z}_1 = z_2\bar{z}_2 = ... = z_n\bar{z}_n = 1$

$\therefore z_1 = \dfrac{1}{\bar{z}_1}, z_2 = \dfrac{1}{\bar{z}_2}, ..., z_n = \dfrac{1}{\bar{z}_n}$

$\therefore |z_1 + z_2 + ... + z_n| = |\overline{z_1 + z_2 + ... + z_n}|$

$$= |\bar{z}_1 + \bar{z}_2 + ... + \bar{z}_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$$

24. (i) (b) Since, $(1+i)z = (1-i)\bar{z}$

$\Rightarrow \dfrac{z}{\bar{z}} = \dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{(1-i)^2}{1-i^2} = \dfrac{1+i^2-2i}{1+1} = -i$

$\Rightarrow z = -i\bar{z}$

(ii) (a) $\because \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iii) (a) Let $z = \dfrac{(2+i)x - i}{4+i} + \dfrac{(1-i)y + 2i}{4i}$

$$= \frac{2x + (x-1)i}{4+i} + \frac{y + (2-y)i}{4i} \times \frac{i}{i}$$

$$= \frac{(2x + (x-1)i)(4-i)}{(4+i)(4-i)} + \frac{-iy + (2-y)}{4}$$

$$= \frac{8x + x - 1 + i(4x - 4 - 2x)}{17} + \frac{(2-y) - iy}{4}$$

$$= \frac{9x - 1 + i(2x-4)}{17} + \frac{2 - y - iy}{4}$$

Now, z is real $\Rightarrow \bar{z} = z$

$\Rightarrow \operatorname{Im} z = 0$

$\Rightarrow \dfrac{2x-4}{17} - \dfrac{y}{4} = 0$

$\Rightarrow 8x - 16 = 17y$

$\Rightarrow 8x - 17y = 16$

(iv) (c) $z = \dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$

$$= \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$= \frac{(3 - 4\sin^2\theta) + i(8\sin\theta)}{1 + 4\sin^2\theta}$$

Since, z is pure imaginary

$\Leftrightarrow \operatorname{Re}(z) = 0$

$\Leftrightarrow \dfrac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} = 0$

$\Leftrightarrow \sin^2\theta = \dfrac{3}{4}$

$\Rightarrow \sin\theta = \pm\dfrac{\sqrt{3}}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}$ $\left(\text{since, } 0 < \theta \le \dfrac{\pi}{2}\right)$

(v) (b) $|z_1 - z_2| = |z_1 + z_2|$

$\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$\Rightarrow z_1\bar{z}_2 = -\bar{z}_1 z_2$

$\Rightarrow \dfrac{z_1}{z_2} = -\dfrac{\bar{z}_1}{\bar{z}_2} = -\left(\overline{\dfrac{z_1}{z_2}}\right)$

$\Rightarrow \dfrac{z_1}{z_2}$ is pure imaginary.